

MACROECONOMETRICS.

Dynamic Regression.

Topic 3:

Recursive VAR models. Exogeneity.

Granger causality.

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- **Exogeneity. Granger causality. Recursive VAR models.**
- Weak Exogeneity.
- Granger Causality.
- Strong Exogeneity.
- Recursive VAR.
- Cholesky decomposition. Structural VAR's.

REFERENCES

- Hendry, 1995, Dynamic econometrics, chapter 5.
- Hendry and Doornik, 2014.
- Enders,W., 2004, Applied Econometric Time Series, Wiley.
- Unpublished typescript notes, “Forecasting with Dynamic Regression Models”.
- Engle et al. (1983), “Exogeneity”, Econometrica, 277-304.

UNIVARIATE ANALYSIS

Espasa ,A., 2005, “Box-Jenkins Analysis” in “An Eponymous Dictionary of Economics :a Guide to Laws and Theorems Named after Economists” Segura,J. y Rodriguez Braun C. (editors).

- Zellner and Palm (1974) the ARIMA model is the final form derived for each endogenous variable in a dynamic multiequation equation model.
- Therefore the use of an ARIMA model for a certain variable X_t is compatible with the fact that X_t is explained in a wider econometric model.

This connection shows the weakness and potential usefulness of ARIMA models in economics.

- **Usefulness**: they provide a formulation of the dynamic properties of the variable which is useful for structural analysis and forecasting.
- This initial knowledge is useful for subsequent econometric study.
- **The limitations** come mainly from the fact that univariate models do not consider relationships between variables. Thus **Granger (2001)** says, 'univariate models are not thought of as relevant models for most important practical purposes in economics, although they are still much used as experimental vehicles to study new models and techniques'.

Example

$$(1 - \Phi_1 L^s - \Phi_2 L^{2s}) \Delta \Delta_s \log X_t = a_t$$

Stationary transformation:

$$\Delta \Delta_s \log x_t = w_t \quad I(2,0)$$

Stochastic systematic growth and seasonality

$$\begin{aligned} \log x_t = & \log x_{t-1} + \left[\log x_{t-s} - \log x_{t-s-1} \right] + & \text{Factors capturing} \\ & + \Phi_1 w_{t-s} + \Phi_2 w_{t-2s} + & \text{the long term} \\ & & \text{Short run oscillations} \\ & & + a_t \text{ Innovations} \end{aligned}$$

EXAMPLE 3

$$\Delta x_t = \sum_{j=1}^s b_j S_j t + (1 - \theta L) a_t$$

Stationary transformation

$$w_t = x_t - x_{t-1} - \sum_{j=1}^s b_j S_j t$$

Calculate $b = 1/s \sum b_j$

Define $b_j^* = b_j - b$ and write
$$\Delta x_t = b + \sum_{j=1}^s b_j^* S_j t + (1 - \theta L) a_t$$

Interpreting the b 's parameters.

$$x_t = x_{t-1} + b + \sum_{j=1}^s b_j^* S_j t +$$

$$-\theta a_{t-1} +$$

$$a_t$$

Non-stationary evolutivity path

Oscillations around the evolutivity path

Innovations

The stationary transformation are the residuals of the regression:

$$\Delta X_t = b + \sum_{j=1}^s b_j^* S_{jt} + W_t$$

Run with the restriction:

$$\sum_{j=1}^s b_j^* = 0$$

- **A VERY SIMPLE
EXAMPLE MOTIVATING
MULTIVARIATE
ANALYSIS**

A VERY SIMPLE EXAMPLE: dividends and earnings.

(see notes)

- Suppose that one is interested in forecasting the **dividend** that some corporation is going to declare next month. If we have a sequence of previous annual dividends denoted by y_t , we could use it to build an ARIMA model and forecast dividends with it.
- However, it would be worthwhile expanding the information set by trying to explain dividends in terms of company **earnings**, after-tax profits, X_t .
- **We have now a system of two equations, being dividends our variable of interest.**

- We then have a **multivariate information set** consisting of present and past values of both variables. Let us denote y_t the variable to be forecast and x_t the related variable added to the information set in order to improve the forecast of y_t .
- This kind of additional variables are called ***explanatory variables***. In the example above, we have only one explanatory variable x_t but, in general, there will be k of these variables denoted by x_{1t}, \dots, x_{kt} .

MEANING OF EXOGENEITY

- In topic 1 we commented that if we have exogenous variables we could **model** without loss of information the remaining ones **conditional on the exogenous variables** (without need to model the latter).
- [*] If all variables are exogenous but one, y_{1t} , then we can work with a **single-equation model** without losing information for the variable of interest.
- **In what follow we assume that [*] is true for our data set.**

- When [*] is not true then the estimation of an equation could require the use of **instrumental variables**.

$$(Y_t/Z_t, X_{t-1}^{+e}; \lambda_1)$$

- The previous breakdown of
- $D(Y_t, Z_t/X_{t-1}^{+e}; \lambda) = D(Y_t/Z_t, X_{t-1}^{+e}; \lambda_1) D(Z_t/X_{t-1}^{+e}; \lambda_2)$
- Can always be done, the question is if from
- $(Y_t/Z_t, X_{t-1}^{+e}; \lambda_1)$ we can model Y_t without losing information.
- That requires that **Z_t to be exogenous**. In particular that
- $D(Y_t/Z_t, X_{t-1}^{+e}; \lambda) = D(Y_t/Z_t, X_{t-1}^{+e}; \lambda_1) D(Z_t/X_{t-1}^{+e}; \lambda_2)$.
- When all variables but one are exogenous, then we have a **single-equation dynamic econometric model**.

DIFFERENT CONCEPTS OF EXOGENEITY

They refer to the different uses that we are going to make of the econometric model.

- 1.- For estimation and inference: **weak exogeneity.**
- 2.- For forecasting: **strong exogeneity.**
- 3.- For simulation **super exogeneity.**

WEAK EXOGENEITY: FOR ESTIMATION AND INFERENCE.

- We need to specify the parameters φ of interest in the estimation or in testing hypotheses.
- $D(Y_t, Z_t / X_{t-1}^{+e}; \lambda) = D(Y_t / Z_t, X_{t-1}^{+e}; \lambda_1) D(Z_t / X_{t-1}^{+e}; \lambda_2).$
- **Two requirements:**
 - - φ must be a function only of λ_1 and
 - - λ_1 and λ_2 have no common elements and are sets of variation free, implying that they are not subject to cross-restrictions. The parameter space for λ is the product of the parameter spaces for λ_1 and λ_2 .

THE INITIAL WORK ON EXOGENEITY

- Koopmans (1950).
- Propose two definitions:
 - - **Predetermined variables** and
 - - exogenous variables (**strictly exogenous variables**).
- A variable in a given equation is predetermined if is independent of the contemporaneous innovation of the equation.
- A variable is strictly exogenous if is independent of the contemporaneous and past innovations of the equation.

PREDETERMINED AND WEAKLY EXOGENOUS VARIABLES

- In linear systems with no restrictions both concepts are the same.

In an unrestricted VAR all the regressors are weakly exogenous and OLS are efficient estimators.

$$\bullet \begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} + \begin{bmatrix} \Phi_{11}(L) & \Phi_{12}(L) & \dots & \Phi_{1n}(L) \\ \Phi_{21}(L) & \Phi_{22}(L) & \dots & \Phi_{2n}(L) \\ \vdots & \vdots & & \vdots \\ \Phi_{n1}(L) & \Phi_{n2}(L) & \dots & \Phi_{nn}(L) \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{nt} \end{bmatrix}$$

EXOGENEITY FOR FORECASTING PURPOSES: STRONG SEASONALITY.

- $D(Y_t, Z_t / X_{t-1}^{+e}; \lambda) = D(Y_t / Z_t, X_{t-1}^{+e}; \lambda_1) D(Z_t / X_{t-1}^{+e}; \lambda_2).$
- A variable Z_t is strong exogenous with respect the parameters of interest if:
 - - it is weakly exogenous and
 - - and the marginal distribution of Z_t does not depend on Y_t .

- Then:
- $D(Y_t, Z_t / X_{t-1}^{+e}; \lambda) = D(Y_t / Z_t, X_{t-1}^{+e}; \lambda_1) D(Z_t / X_{t-1}^{+e}; \lambda_2)$
- and we say that present and past values of Y_t do not determine Z_t .
- This means that Y_t does not Granger cause Z_t .

STRONG EXOGENEITY

Two conditions:

- Is weakly exogenous and
- There is no Granger causality from the endogenous variable to the exogenous.

STRICTLY AND WEAKLY EXOGENOUS VARIABLES

- In linear systems with no restrictions both concepts are the same.

SUPEREXOGEITY

- This concept is required for simulation and control and it is a condition for invalidating the Lucas critique.
- The Lucas critique says that is inappropriate to use a econometric model for simulating or evaluating new policy measures if with the new policy the model changes.

CONDITIONS FOR SUPEREXOGENEITY

- - The variable is weakly exogenous, implying that the parameters of interest are only functions of Λ_1 and
- Λ_1 is invariant under changes in the marginal distribution of the Z's.

• GRANGER CAUSALITY

REFERENCES

- Enders, 2004, section 5.8.

GRANGER CAUSALITY IN STATIONARY MODELS

A bivariate system: 2 variables (y, z) ,

variable y does not Granger-cause variable z

If for every $s > 0$, Mean Squared Error (MSE) of the forecast of z_{t+s} given (z_1, \dots, z_t) is the same as the MSE of the forecast of z_{t+s} given $(y_1, \dots, y_t, z_1, \dots, z_t)$.

Testing for Granger Causality by means of a F-test.

$$Z_t = c + \alpha_1 z_{t-1} + \dots + \alpha_p z_{t-p} + \beta_1 y_{t-1} + \dots + \beta_p y_{t-p} + a_t$$

$$H_0 : \beta_1 = \dots = \beta_p = 0. \quad (30)$$

If you do not reject H_0 then we say that y does not Granger-cause z .

- **POSSIBLE OUTCOMES**

- y does not causes z .
- y causes z in the Granger's sense.

Granger causality can be tested in both directions

- Causality **from y to z**; it can be tested in regression (30) and
- Causality **from z to y**. It can be tested in regression (31).

$$Y_t = c + \alpha_1 Y_{t-1} + \dots + \alpha_p Y_{t-p} + \beta_1 Z_{t-1} + \dots + \beta_p Z_{t-p} + a_t \quad y$$

$$H_0 : \beta_1 = \dots = \beta_p = 0. \quad (31)$$

POSSIBLE RESULTS WHEN TESTING CAUSALITY BETWEEN TWO VARIABLES

-(A) **Bidirectional causality**: you reject H_0 in (30) and (31).

$$Z_t = c + \alpha_1 z_{t-1} + \dots + \alpha_p z_{t-p} + \beta_1 y_{t-1} + \dots + \beta_p y_{t-p} + a_t \quad (30)$$

$$Y_t = c + \alpha_1 Y_{t-1} + \dots + \alpha_p Y_{t-p} + \beta_1 Z_{t-1} + \dots + \beta_p Z_{t-p} + a_t \quad (31)$$

-(B) **Unidirectional causality from y to z**: you reject the null in (30) but not in (31).

-(C) **Unidirectional causality from z to y**: you reject the null in (31) but not in (30).

-(D) **No causality**. You do not reject the null in (30) and (31).

Selection of lags

- To test for Granger causality you need to specify the number of lags and you can do it applying the AIC or other information criteria.

A VAR model with a unidirectional Granger causality between two variables.

VAR WITH TRIANGULAR DYNAMIC STRUCTURE

EXAMPLE: VAR (1) for two variables (y_1, y_2)

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

Because $\Phi_{12} = 0$ y_2 does not caused y_1 .

- **EXTENSION TO MORE
THAN TWO VARIABLES**

LAG SELECTION

- It can be done by the multivariate AIC
- $AIC = T \log \det(\Omega) + 2N$,
- T are the number of observations,
- Ω is the residual variance-covariance matrix
- N is the total number of parameters estimated in all the equations.

$$\begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} + \begin{bmatrix} \Phi_{11}(L) & \Phi_{12}(L) & \dots & \Phi_{1n}(L) \\ \Phi_{21}(L) & \Phi_{22}(L) & \dots & \Phi_{2n}(L) \\ \vdots & \vdots & & \vdots \\ \Phi_{n1}(L) & \Phi_{n2}(L) & \dots & \Phi_{nn}(L) \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{nt} \end{bmatrix}$$

C_j : are constants

$\Phi_{ij}(L)$: is the dynamic polynomial of variable X_j in the equation for variable i .

An $\Phi_{ij}(L)$ of order p can be written as:

$$\Phi_{ij}(L) = \Phi_{ij}^{(1)}L + \Phi_{ij}^{(2)}L^2 + \dots + \Phi_{ij}^{(p)}L^p$$

Therefore $\Phi_{ij}^{(r)}, r = 1, \dots, p$ are the coefficients of polynomials Φ_{ij}

- In the general stationary VAR we say that variable X_j does not Granger cause variable X_i if we don't reject the hypothesis that all the coefficients in $\Phi_{ij}(L)$ are zero.

BLOCK TESTING

- Variables $r, r+1, \dots, n$ will be not be Granger caused by variables $1, 2, \dots, r-1$ if polynomials $\Phi_{ij}(L)$ for $i = r+1, r+2, \dots, n$ and $j = 1, 2, \dots, r-1$ are zero.
- This can be tested by the likelihood ratio test
- $\Lambda = T [\log \det(\Omega_r) - \log \det(\Omega_u)]$
- Or by the multivariate AIC

SURE and VAR models

- VAR is a system with N multiple regressions.
- SURE is a system Seemingly Unrelated Regression Equations:
- Is a system of r variables, y_1, \dots, y_r , in which any of them depend on k regressors z_1, \dots, z_k , but do not depend on the other y 's.
- Nevertheless if the residual variance-covariance matrix is not diagonal the y 's are contemporaneously related.

EXAMPLE OF A THREE-EQUATION SURE MODEL:

$$y_{1t} = C_1 + a_1X_{1t} + a_2X_{2t} + a_3X_{3t} + e_{1t} \quad (1)$$

$$Y_{2t} = C_2 + b_1X_{3t} + b_2X_{4t} + e_{2t} \quad (2)$$

$$Y_{3t} = C_3 + d_1X_{1t} + d_2X_{5t} + e_{3t} \quad (3)$$

$c's$, $a's$, $b's$ and $d's$ are coefficients and

$$Var \begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$$

SINCE THE RESIDUALS ARE CONTEMPORANEOUSLY CORRELATED SO ARE THE $y's$.

EFFICIENT ESTIMATION REQUIRE GLS.

ESTIMATING SURE MODELS

- The efficient estimation of the system requires GLS, unless all the regressors are the same in all the equations.
- This is the case of VAR models, which can be efficiently estimated by OLS.

Interpretation of Granger causality

- Granger causality must be interpreted in terms of helping to improve the forecast, but
- Not in terms of real causality.

Weak exogeneity and Granger causality

- W.E. implies that explanatory variable is not contemporaneously related with the dependent variable.
- The lack of Granger causality refers to the fact that lags of the explanatory variable do not determine the dependent variable.

TRIANGULAR VAR

- $$\begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} + \begin{bmatrix} \Phi_{11} & 0 & \dots & 0 \\ \Phi_{21}(L) & \Phi_{22}(L) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \Phi_{n1}(L) & \Phi_{n2}(L) & \dots & \Phi_{nn}(L) \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{nt} \end{bmatrix}$$
-

TRIANGULAR VAR

- Unidirectional causality from X_{jt} to $X_{(j+r)t}$, for
- r and $j > 0$.

$$\bullet \begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} + \begin{bmatrix} \Phi_{11} & 0 & \dots & 0 \\ \Phi_{21}(L) & \Phi_{22}(L) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \Phi_{n1}(L) & \Phi_{n2}(L) & \dots & \Phi_{nn}(L) \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{nt} \end{bmatrix}$$

Example

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

Being $\Phi_{12} = 0$ y_2 does not Granger cause y_1 .

• RECURSIVE VAR

RECURSIVE VAR MODEL

- We say that a VAR is recursive if:
- (a) if it is possible to order the variables such that the dynamic polynomial matrix has a triangular structure and
- (b) and the residual variance-covariance matrix is diagonal..

PROPERTIES OF THE RECURSIVE VAR MODELS

- All the explanatory variables in all equations are strongly exogenous.
- The model can be efficiently estimated by **OLS**.
- **The forecast of each variable** can be done from a single-equation system –like the corresponding equation in the VAR-conditional to forecasts of the explanatory variables obtained outside of this system.

Triangular dynamics with contemporaneous dependency.

$$x_t = \Phi_{11} x_{t-1} + a_{1t} \quad (1.a)$$

$$y_t = \Phi_{21} x_{t-1} + \Phi_{22} y_{t-1} + a_{2t} \quad (1.b)$$

$$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

**Orthogonalizing the
residuals according
with a triangular
structure (Choleski
decomposition).**

Orthogonalization of the residuals

- 1.- Run a simple regression between the error term in the equation of interest (a_{2t}) on the other error term:

- $a_{2t} = ba_{1t} + \varepsilon_t \quad (2)$

In (2) ε_t and a_{1t} are orthogonal.

- 2.- Put a_{1t} in terms of the observed data as

$$a_{2t} = bx_t - b\Phi_{11} x_{t-1} + \varepsilon_t \quad (3)$$

New equation for the variable of interest

- 3.- In the equation for the variable of interest substitute the residual term by its value in (3)
- $y_t = bx_t + (\Phi_{21} - b\Phi_{11}) x_{t-1} + \Phi_{22} y_{t-1} + \varepsilon_t$
(4)
- $y_t = bx_t + b_1 x_{t-1} + \Phi_{22} y_{t-1} + \varepsilon_t.$ (5)

The resulting recursive VAR

- $x_t = \Phi_{11} x_{t-1} + a_{1t}$ (6.1)

- $y_t = b x_t + b_1 x_{t-1} + \Phi_{22} y_{t-1} + \varepsilon_t$ (6.2)

- It has triangular dynamic structure and
- The error terms a_{1t} and ε_t are uncorrelated.

This results from the fact that we have imposed a triangular structure in the causality between the residuals: **the contemporaneous shocks of X_t (a_{2t}) affect X_t and Y_t , the shocks of Y_t only to Y_t .**

We can denote σ^2 to the variance of ε_t in the equation (6.2).

Then

$$\sigma_2^2 = \beta^2 \sigma_1^2 + \sigma^2 \quad (7)$$

and β is

$$\beta = \rho \frac{\sigma_2}{\sigma_1}, \quad (8)$$

where ρ is the correlation ($\sigma_{12}/\sigma_1\sigma_2$) between a_{1t} and a_{2t} .

Therefore

and

$$\sigma_2^2 = \rho^2 \sigma_2^2 + \sigma^2 \quad \sigma^2 = \sigma_2^2 (1 - \rho^2) \quad (9)$$

The single-equation econometric models –dynamic regression models- can include as regressors the contemporaneous values of the explanatory variables.

GENERALIZATION OF THE CHOLESKI DECOMPOSITION TO MORE VARIABLES

VAR (1):

$$\underline{X}_t = \Phi_1 X_{t-1} + \underline{a}_t \quad (1)$$

$$Var(a_t) = \Omega$$

STRUCTURAL VAR:

$$Bx_t = B\Phi_1 x_{t-1} + B\underline{a}_t \quad (2)$$

$$Bx_t = B\Phi_1 x_{t-1} + e_t \quad (3)$$

It is not identified: the contemporaneous dependency is in both, B and Ω .

We need to restrict, for instance matrix B (**Simultaneous Econometric Models**).

Choleski restriction:

Note $Var(a_t) = \Omega$ and $Var(e_t) = D$ (diagonal matrix)

$$a_t = B^{-1}e_t$$
$$\Omega = B^{-1}D(B^{-1})'$$

$$\boxed{B\Omega B' = D}$$

TRIANGULAR STRUCTURE IN THE CAUSALITY BETWEEN THE RESIDUALS.

- $a_t = B^{-1}e_t$
- $\Omega = B^{-1}D(B^{-1})'$

$$\boxed{B\Omega B' = D}$$

Being Ω a symmetric positive definite matrix ,
there exists a triangular matrix B which
transforms Ω in a diagonal matrix (D).

STRUCTURAL VAR

- The Choleski distribution can be applied to any type of VAR

TESTING IN VAR SYSTEMS

- See Enders p.285.
- The econometric models can include stationary and non-stationary variables.
- In a previous course on univariate analysis you studied that the t-test corresponding to the parameter that is a unit root, has not the standard distribution. And the distribution must be simulated as Dickey and Fuller did initially.

TESTING IN MODELS WITH STATIONARY AND NON-STATIONARY VARIABLES.

- Sims, Stock and Watson (1990):
- “If the coefficient of interest can be written as a coefficient on a stationary variable, then the t-test is appropriate”.
- No problem in testing if stationary variables matters in equation for non-stationary variables.
- The usefulness of the Equilibrium Correction Model.

Exogeneity condition for Non-stationary variables

- Write the model as a VEqCM.
- For a variable to be exogenous the cointegration restrictions can not enter in its equation.

The Vector Equilibrium Correction Model

$$\Delta X_t = \mu + \alpha\beta'X_{t-1} + \Gamma_1\Delta X_{t-1} + \dots + \Gamma_{k-1}\Delta X_{t-k+1} + \epsilon_t \quad (2)$$

- The $\beta'X_{t-1}$ is a vector of r cointegration relationships, which captures the long-run relationships (restrictions) between the variables.
- Being X_t non-stationary, $\beta'X_{t-1}$ is stationary.
- α is a $n \times r$ matrix of adjustment coefficients.
- $\beta'X_{t-1}$ are deviations from the equilibrium relations, thus **VEqCM**.