

Dynamic Regression.

Topics 5 and 6:

Cointegration. Error correction models. Spurious correlation. Testing for cointegration.

Antoni Espasa

Tor Vergata

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antoni.espasa@uc3m.es

6th and 10th May 2016: Topic 5

- **Cointegration. Error correction models.**
- The cointegration in the simile of pearls by Clive Granger (2003).
- The cointegration, definition and examples.
- The CVAR as a reduced rank restriction in a VAR system.
- The error correction model.
- Deterministic elements in the error correction model.
- Cointegration in ADL models.
- Examples.

REFERENCES

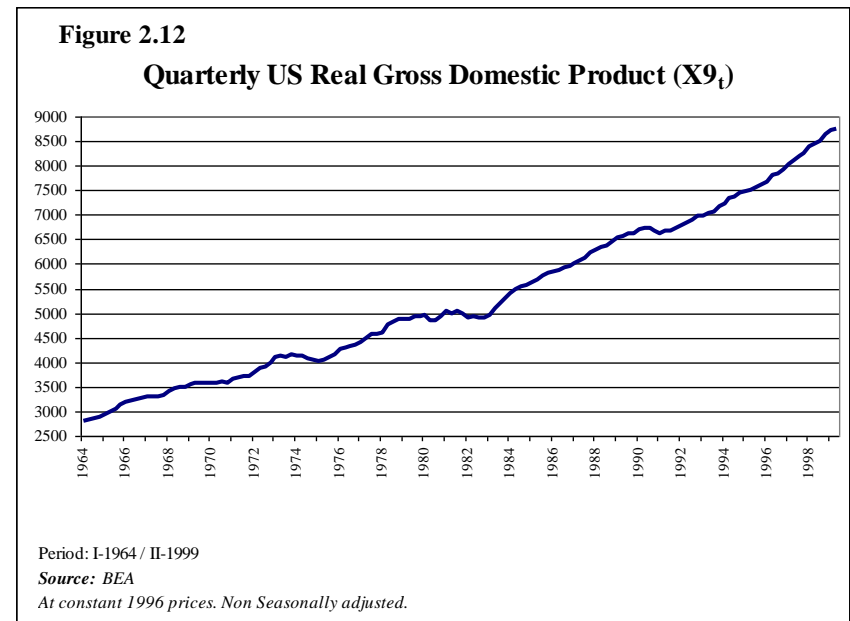
- “Time series analysis, cointegration and applications”, Nobel lecture, C: Granger, December 8, 2003.
- Juselius, K. (2014) “Haavelmo’s probability approach and the cointegrated VAR”, *Econometric Theory*, doi. 10.1017/S0266466614000279.
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Integrated time series

Granger simile

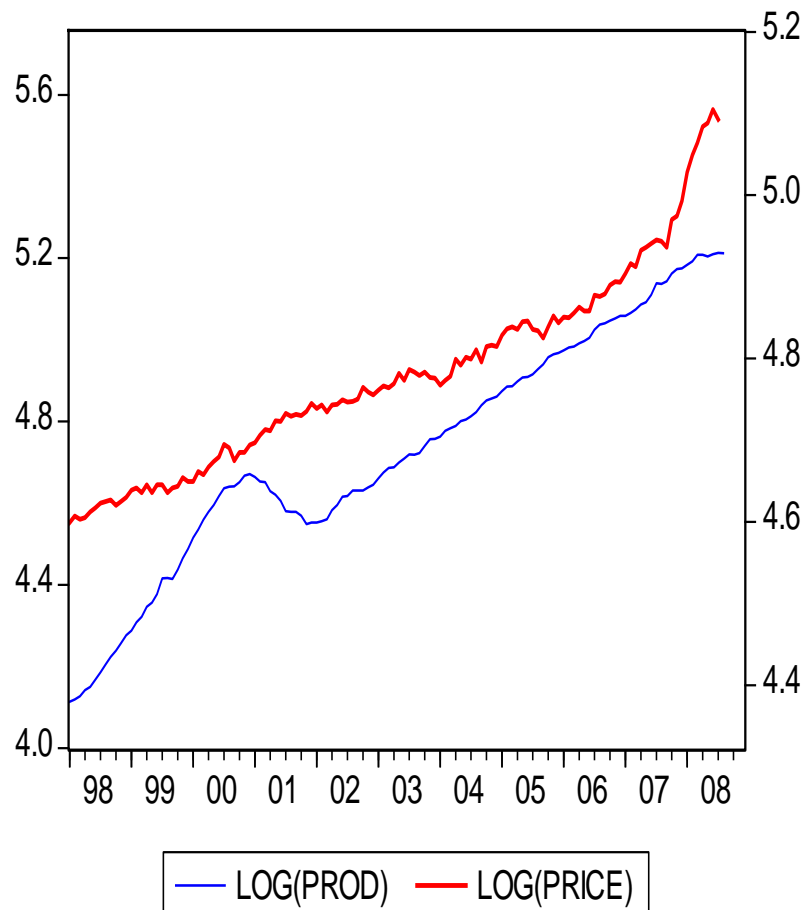
- Suppose that you have a loosely strung string of pearls which you throw down, gently, onto a table top with the string of pearls roughly stretched out.
- **You will have created a time series**, with time represented by the distance down the table, the size of value of the variable represented by the distance from the edge of the table to a point and the pearls representing the points in the series.

- As the position of one pearl will have an impact on the position of the next, since they are strung together, this series will appear to be **rather smooth**, with no large fluctuations in value for one term to the next.



TWO INTEGRATED SERIES

- We are asked to suppose that we have **two similar strings of pearls**, which we throw down onto the table separately —it is also assumed that the strings do not cross each other.



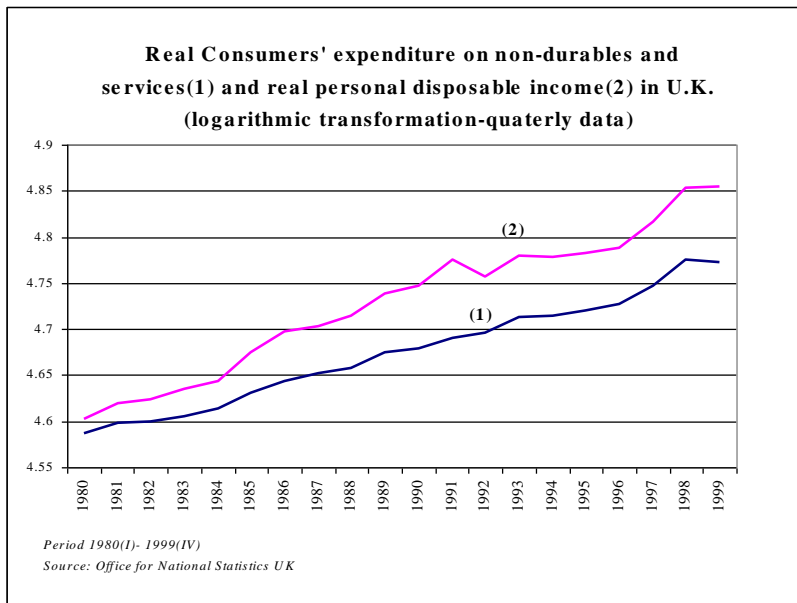
- The result would be **two smooth** (integrated) series with different and unrelated evolutions and, if we calculate **the distance** between the two series and represent it graphically, **it would also be a smooth (integrated) time series.**

To visualise a pair of cointegrated series

- If the pearls had been strung with small but powerful magnets, the two strings might have attracted each other when thrown onto the table, resulting in **two series with similar smooth but not identical forms.**

-

Figure 2.13



- Now, according to Granger, the distance between the two strings would be a stationary series and this would be an example of cointegration.

- In other words, he continues, cointegration refers to the possibility of **two integrated** (smooth) **series**, possibly rescaled, **evolving in similar but not identical ways** **with the difference between them being stationary.**

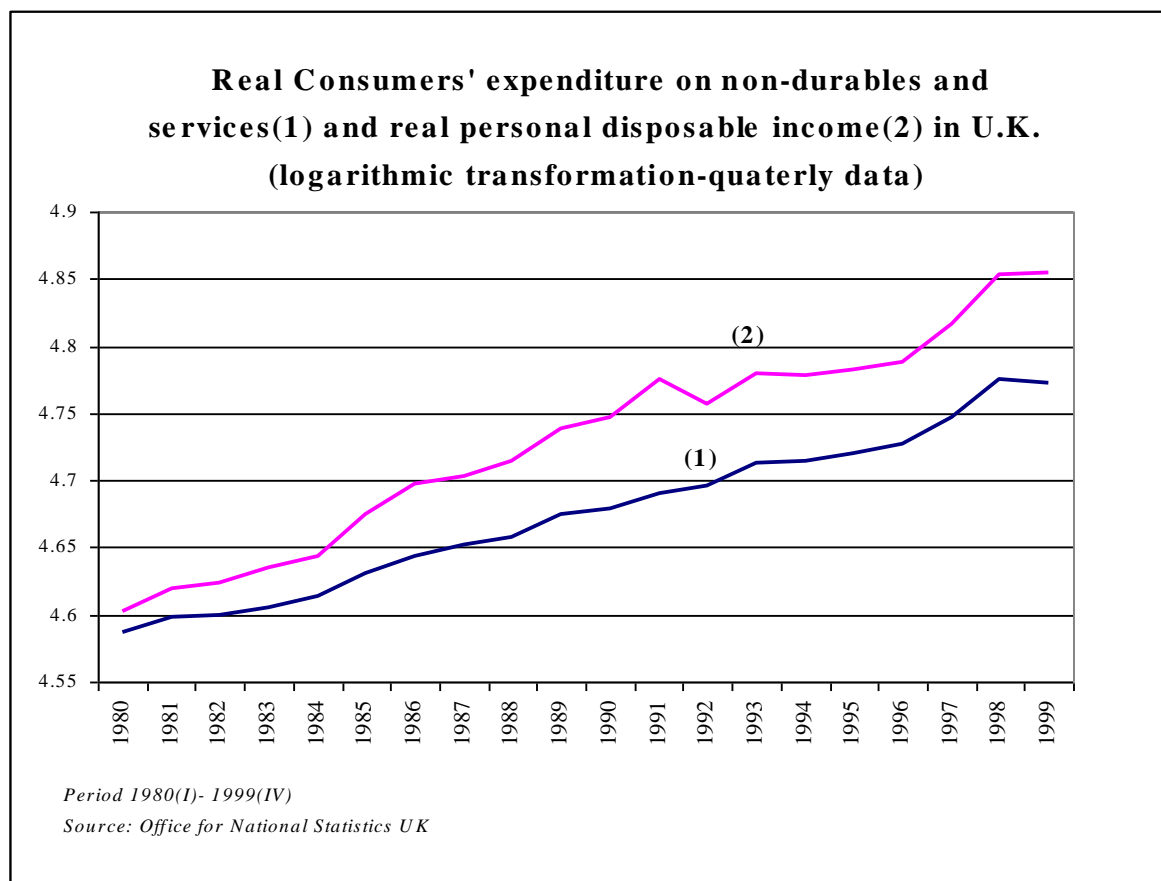
This stationary difference between two cointegrated series

- is a restriction to their smooth long-term evolutions.
- In sum, the property of **cointegration** — the difference between two integrated series is stationary — **implies (1)** the existence of a **long term equilibrium relation between the two series**, and **(2)** the difference between them is, in itself, the **short term disequilibrium between the two series**

Example.

- The average propensity to consume — the consumption logarithm less the income logarithm — usually is a stationary series, showing that:
- in the long term, there is an equilibrium relation consisting of the fact the consumption is in proportion to income.
- .

Figure 2.13



- The cointegration theory has therefore enabled us to **unify economic theory, like the consumption theory, with the construction of dynamic macroeconomic models.**
- **Hendry (2004b)** describes Clive Granger's contribution as “an extraordinary breakthrough in the empirical analysis of macroeconomic relations and in testing macroeconomic theories”, **extending Haavelmo's results to integrated economic series.**

Integrating factors

- After an integrated series, we can conceive the existence of a hidden factor determining this property of integration.
- In the pearl example, this factor would be the throwing down of the extended string of pearls.
- TWO TIME SERIES. When throwing two strings, each one would receive different impulses and we would have two different hidden factors, configuring two independent smooth evolutions

Integrating factors in cointegrating variables

However, **if the pearls are strung with magnets**, they will cancel the differences between the hidden factors in each series, reducing them to **a single factor** and therefore obtaining similar smooth evolutions.

In this case, the trends (smoothness of evolution) or long term in the two cointegrated series are generated by a single linking factor which determines that **such trends are not independent but restricted by an equilibrium relation**.

PREVIOUS STUDIES ON EQUILIBRIUM CORRECTION MODELS

- In **the equilibrium correction model** -considering the simplest case of two variables- the increment of a time series is explained according to:
 - a constant,
 - its previous increments and those of the other variable, but also – and this is the fundamental aspect of these models –
 - according to the value at the previous moment of the difference between the two series.
-

Sargan (1964)

- In the example presented by Sargan (1964), this difference (on the log transformations of the series) directly included the position of disequilibrium of the variables before the present time.

The long run in the equilibrium correction model.

- In the long run, a stationary element tends towards its mean, and so do variable increments, so in the long run, Sargan's equilibrium correction models only show that the difference between the logs of the variables is a constant.

- In other words, in the long run, the level of one variable is proportional to the level of the other.
- This model, therefore, represents a long term equilibrium relation between two levels of the variables within an equation which is apparently in increments (first differences).

- Models with equilibrium correction mechanisms were developed and applied in different papers by David Hendry and his associates, such as *Davidson et al. (1978)*, showing their success in modelling non-stationary series.
- There are **therefore other ways to obtain stationary transformations besides mere differentiation.**

Untested formulation of the EqCM

- However, as **Granger pointed out** in his remarks on these papers, **models with equilibrium correction mechanisms** were formulated by **including untested theoretical results** concerning the stationarity of the difference between two integrated variables, implying a linear combination between them in which the coefficients were established *a priori*, plus one and minus one.

There was a need to contemplate the overall process generating all the variables (system) and test whether it gave rise to a model with an equilibrium correction mechanism.

- That was how intending to demonstrate that the EqCM was wrong Granger (1981) discovered cointegration.
- Suppose that we have a system of two variables, X and Y , and that they are both $I(1)$. Granger's work (1981) aimed at the issue of, if the dependent variable of a model was $I(0)$ — because it was the increment of a variable $I(1)$ as variable Y , for instance, — it could be represented in terms of variables $I(1)$ like variables X and Y on levels, as was the case in models with equilibrium correction mechanisms.
- Finally Clive Granger shows that it is possible.

Cointegration and spectral analysis

- The discovery of cointegration, which establishes the conditions under which, in a system of integrated variables $I(1)$, it is possible to explain a stationary variable $I(0)$ — in terms of the variables $I(1)$, occurred using spectral analysis.
- Here, the concept of correlation (squared) can refer to all frequencies and is called coherence. Cointegration implies that coherence on frequency zero (band of frequency I) is one. In other words, **there is perfect correlation in the long run although the series contain very different short term fluctuations.**
-

THE DIMENSION OF A COINTEGRATED SYSTEM IN THE LONG RUN

- In the long term, the dimension of a bivariate system is not two, but reduces to one due to the perfect correlation between the variables on the long term band of frequencies.
- **This relation cannot be imposed *a priori* in the formulation of a model for one variable in terms of the other, but we have to formulate the system of both variables and test that it meets the cointegration restriction.**

When there is cointegration,

- the equilibrium correction mechanism model is derived from the system representing the two variables and
- is no longer taken *a priori* without testing its validity.

- A **bivariate system** relating consumption and income is normally used to **deduce an equilibrium correction model between the two variables**,
- thus showing that, in the long run, the difference (in the log transformations) between consumption and income is stationary.

As Granger said in his lecture,

- the equilibrium correction model “has been particularly important in making the idea of cointegration practically useful”.

Cointegration.

Definition.Examples.

The Granger example

- was referred to two integrated variables $I(1)$ - such that rescaling them the difference:
- $Y_t - (\beta X_t + c)$
- is stationary.
- It means that Y and X are non-stationary but a linear relationship between them is.

GENERAL DEFINITION COINTEGRACIÓN

Given a vector of variables $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})'$

We will say that their elements are cointegrated if:

- (1) All the elements are integrated of the same order $d, I(d)$ and
- (2) There exists at least a linear combination between them

$$\beta_1 x_{1t} + \beta_2 x_{2t} + \dots + \beta_n x_{nt}$$

of a lower order of integration $(d-b)$, $b > 0$, $I(d-b)$.

The vector $\beta = (\beta_1, \beta_2, \dots, \beta_n)'$

Is called **the cointegration vector** and we say that the **variables are cointegrated of order $CI(d,b)$** .

Most of the theory and applications refer to
cointegración $CI(1,1)$:

The variables are $I(1)$ and there exists at least
a linear combination between them which is
stationary.

COINTEGRATION AND LONG RUN EQUILIBRIUM BETWEEN VARIABLES

In $CI(1,1)$ there is at least a linear combination

$$\beta_1 x_{1t} + \beta_2 x_{2t} + \dots + \beta_n x_{nt} + c = m_t$$

$$\beta' x_t = m_t$$

which is stationary. In the long run m_t tend to its zero mean.

Therefore $\beta' x_t = 0$ is a long run equilibrium relationship.

The long run of cointegrated variables are not independent but restricted by: $\beta' x_t = 0$.

At any moment in time $\beta' x_t = m_t$ will not be zero but stationary, representing the **deviation from equilibrium**.

EXAMPLES OF LONG RUN EQUILIBRIUM RELATIONSHIPS BETWEEN ECONOMIC VARIABLES.

(a) Relationship between spot s_t , and forward, f_t , commodity prices in efficient markets. In these cases s_t y f_t are $I(1)$

$$f_t - s_t$$

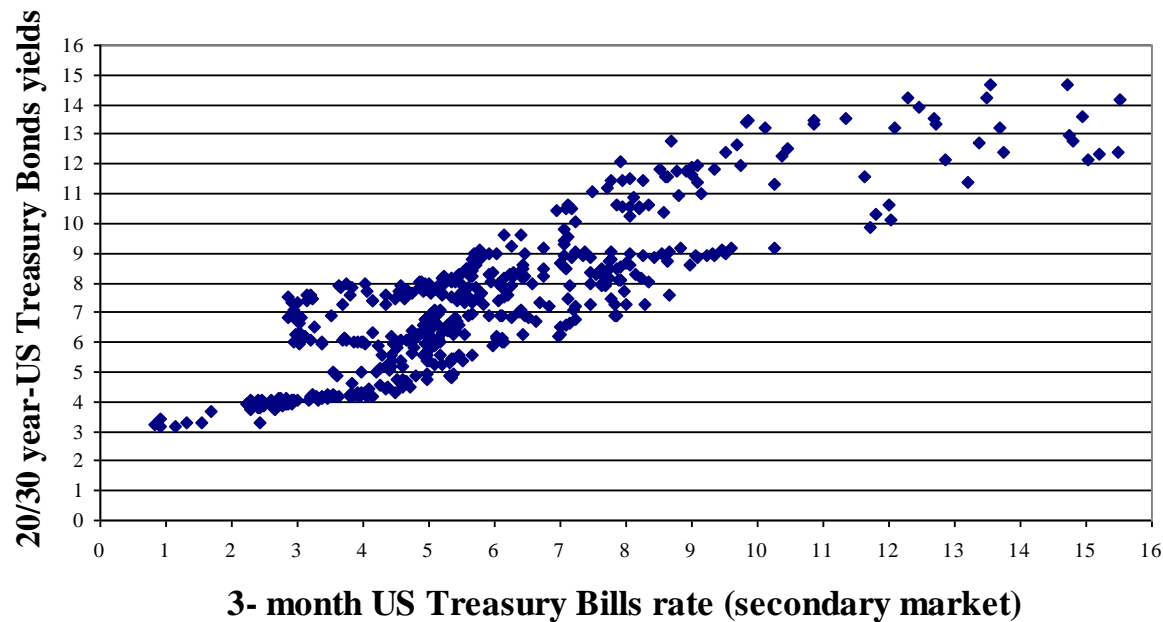
Is stationary.

The long run equilibrium relationship is

$$f_t = s_t$$

SHORT AND LONG RUN INTEREST RATES.

The spread is stationary.



Period:1958.01-2000.01

Source: Federal Reserve Board of Governors

CONSUMPTION AND INCOME.

C_t and Y_t , are $I(1)$ but

$$\log C_t - \beta - \log Y_t \quad ,$$

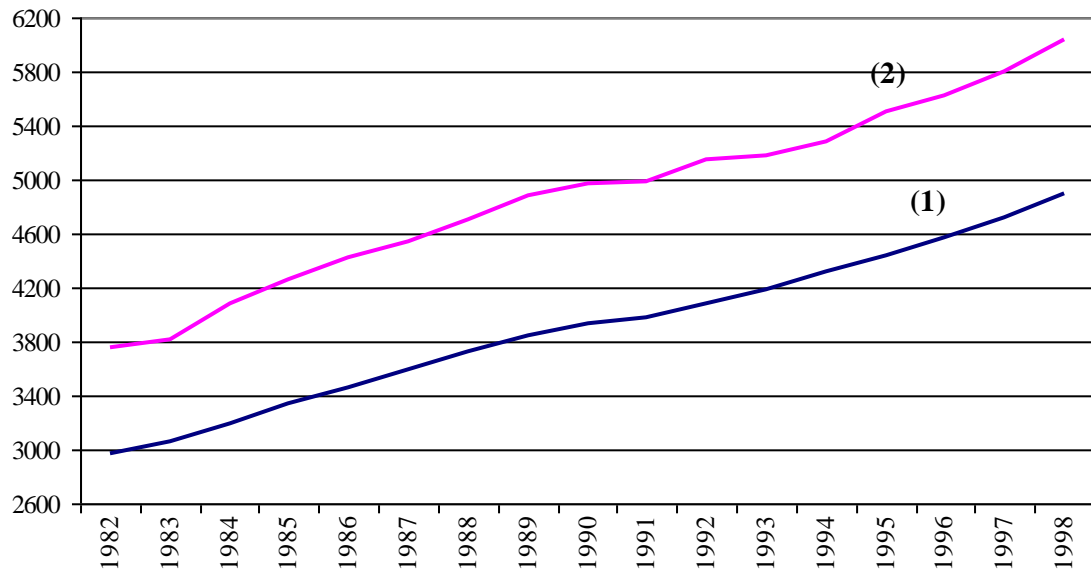
is stationary.

See Hendry and Doornik, 2014, chapter 16.

CONSUMPTION AND INCOME IN US

Figure 2.13

**Real Consumers' expenditure on non-durables and services(1)
and real personal disposable income(2) in U.S.**



Source: Department of Commerce US. BEA

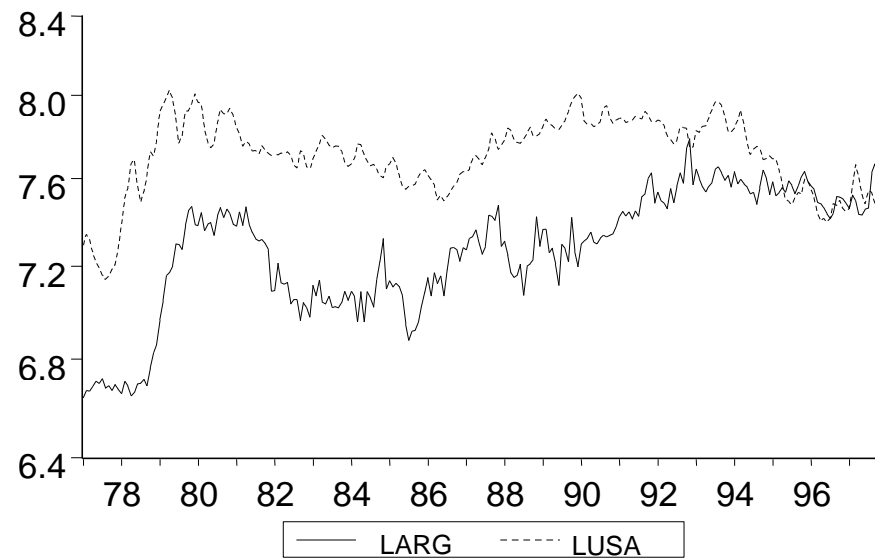
EXAMPLES OF LONG RUN EQUILIBRIUM RELATIONSHIPS

(d) Arbitrage in the prices of similar goods
in different markets.

In the long run

$$P_{it} - P_{jt}$$

would be zero.



(f) Very often, imports, M_t , GDP, Y_t , and a relative prices, PR_t , are $I(1)$ but there exists a linear relationship

$$M_t - \beta_1 Y_t - \beta_2 PR_t ,$$

Which is stationary.

MONEY DEMAND

- Money
- Prices
- Income
- Interest rates.
- See Hendry and Nielsen (2007), chapter 17.

- **THE COINTEGRATED
VAR MODEL,**
- **CVAR**

VAR: the matrix Π

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \dots + \Gamma_{p-1} \Delta x_{t-p+1} + a_t \quad (2)$$

$$\Gamma_i = -(\Phi_{i-1} + \dots + \Phi_p)$$

$$i = 1, \dots, p-1$$

$$\Pi = -(I - \Phi_1 \dots - \Phi_p)$$

The Cointegrated VAR model as a general structure for macro-econometric dynamic models.

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \dots + \Gamma_{p-1} \Delta x_{t-p+1} + a_t$$

Three cases:

$\Pi = 0$: A **Var on first differences**, variables are $I(1)$. $r(\Pi)=0$

Π is invertible: A **Var in levels**; variables are $I(0)$. **Full rank**

Π is not invertible; A **CVAR**; variables are $I(1)$ and cointegrated. **Intermediate rank**

Alternatives in the rank of Π

The general model (G) after reduction process

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \dots + \Gamma_{p-1} \Delta x_{t-p+1} + a_t$$

The term Πx_{t-1} captures the long term,
all the other elements are stationary, non-persistence.

$r = 0$ the model is a VAR on first differences.

(no restrictions in the long run)

$R = n$ the model is a VAR on levels. (Variables are stationary)

$0 < r < n$ the long run is restricted. A restricted VAR in Π .

This VAR can be formulated in terms of the
equilibrium correction mechanism.

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COINTEGRATION

Π has reduced rank, then:

$$\Pi_{n \times n} = -\alpha_{n \times r} \beta'_{r \times n}$$

$$\text{rank}(\alpha) = \text{rank}(\beta) = r$$

(r independent cointegration relationships)

THE CONCEPT OF COINTEGRATION BETWEEN I(1) VARIABLES

- A set of $X_{1t}, X_{2t}, \dots, X_{nt}$ which are integrated of order I(1) are cointegrated if there is at least a linear restriction between them which is stationary:
- $m_t = c + \beta_1 X_{1t} + \beta_2 X_{2t} + \dots + \beta_n X_{nt} \quad (A)$
- The expected value of m_t is zero therefore in the long run:
- $-\beta_1 X_{1t} = c + \beta_2 X_{2t} + \dots + \beta_n X_{nt} \quad \text{or}$
- $X_{1t} = c^* + b_2 X_{2t} + \dots + b_n X_{nt} \quad (B)$

- $-\beta_1 X_{1t} = c + \beta_2 X_{2t} + \dots + \beta_n X_{nt}$ or
 - $X_{1t} = c^* + b_2 X_{2t} + \dots + b_n X_{nt}$ (B)
 - (B) is an equilibrium relationship.
-
- $m_t = c + \beta_1 X_{1t} + \beta_2 X_{2t} + \dots + \beta_n X_{nt}$ (A)
 - (A) is the stationary deviation from the above equilibrium equation.

Reduced rank of matrix Π

$$X_t = \mu + \Phi_1 X_{t-1} + \dots + \Phi_k X_{t-k} + \epsilon_t \quad (1)$$

$$\Delta X_t = \mu + \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1} \Delta X_{t-k+1} + \epsilon_t \quad (2)$$

$$\Pi = \alpha \beta'$$

NUMBER OF COINTEGRATION RELATIONSHIPS

- In a vector on n variables, the matrix Π is $n \times n$.
- The maximum number of **independent** cr's can be $(n-1)$.
- If there were n the system would be stationary.
- When we have $r > 1$ cr's, then β is matrix $n \times r$ and its columns r the cointegrated vectors.

THE COINTEGRATING VECTORS ARE CANONICAL

- They are invariant to increasing the information set.
 - 1.- Adding more variables the original cr's are preserved.
 - 2.- Aggregating along time does not remove cointegration.

Adding more variables in a CVAR

Could lead to more cr's.

In this case the adjustment α and transitory dynamic Γ coefficients could change.

Economic interpretation requires the inclusion of all relevant variables.

The empirical cointegration analysis could be unreliable when the number of variables is moderately large.

Use the mentioned canonical property and after find cointegration in a initial set of variables try the extend the set.

Multiple forms of the cointegration relationships

If $\beta'x_t = m_t$ is stationary ,

- $c \times m_t$ is also stationary.
- **In practice we normalized.** With just one cr, normalization is done by assigning unit coefficient one of the variables, such that the resulting long term relationship could have economic meaning.

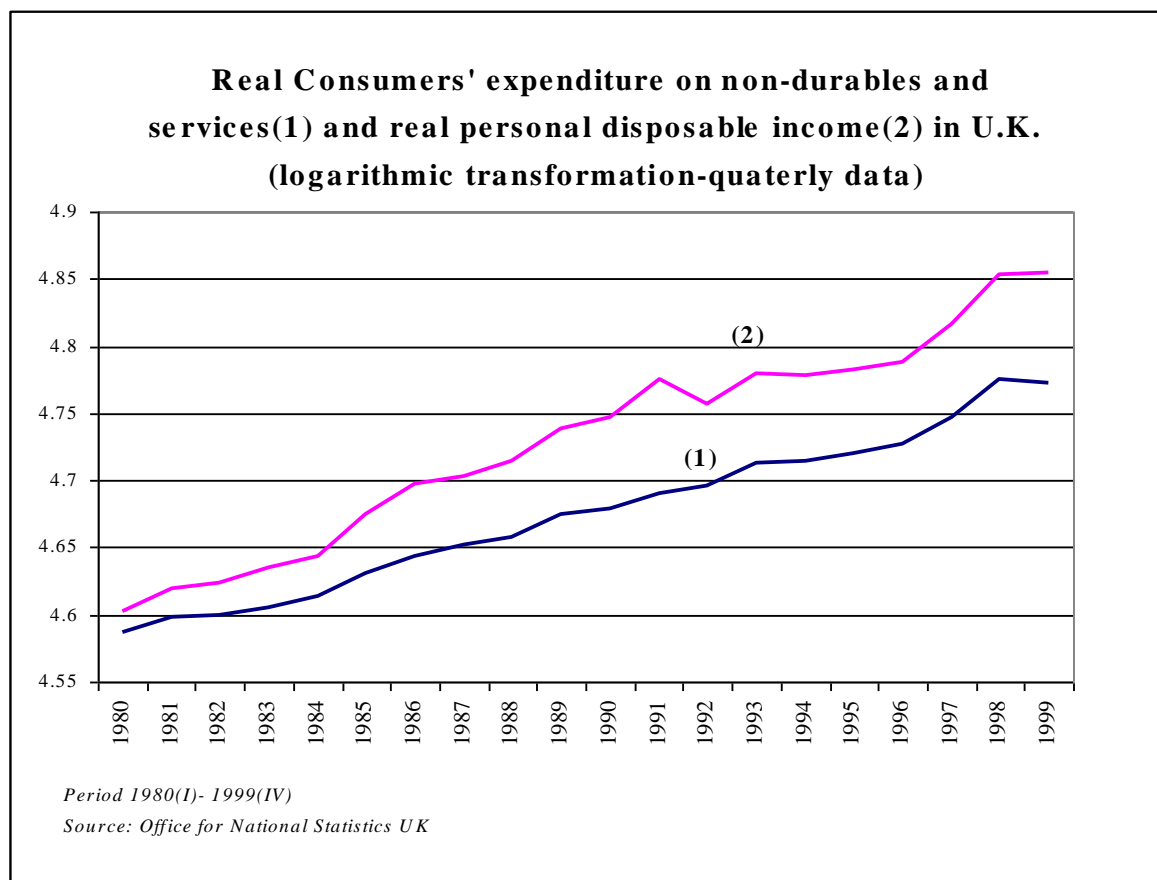
Normalization in the case of several cr's.

- This can be done assigning a unit coefficients to different variables in the r cointegration relationships.

Multicollinearity in I(1) variables and cointegration.

- Multicollinearity refers to the case in which the sample correlation between two variables is close to one.
- Two variables I(1) usually will show multicollinearity, because even when they are independent both show a smooth evolution as pointed by Granger.
- In a CVAR the variables enter on differences or as $\beta'x_t$, which are stationary transformations. Also the collinearity between Δx_t and βx_t would be, in general small.

Figure 2.13



Inference in cointegrated systems

- It can be shown that in a regression between two cointegrated variables **the estimators are superconsistent**: Converge at $1/T$.
- But the **inference can't be done by the standard procedures**. The asymptotic distribution of the statistics are not the standard ones and they must be obtained by simulation.

TESTING HYPOTHESIS IN NON-STATIONARY MODELS

- Sims, Stock and Watson, Econometrica 1990, show that:
- *If the coefficient on which we want to test a certain hypothesis can be written as the coefficient of a stationary variable, then the t statistic has the usual t distribution.*
- Application in Granger causality.

The Cointegrated VAR model as a general structure for macro-econometric dynamic models.

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \dots + \Gamma_{p-1} \Delta x_{t-p+1} + a_t$$

The Vector Equilibrium Correction Model

As a general model after the reduction process.

$$\Delta X_t = \mu + \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1} \Delta X_{t-k+1} + \epsilon_t \quad (2)$$

- The $\beta'X_{t-1}$ is a vector of r cointegration relationships, which captures the long-run relationships (restrictions) between the variables.
- **Being X_t non-stationary, $\beta'X_{t-1}$ is stationary.**
- α is a $n \times r$ matrix of adjustment coefficients.
- $\beta'X_{t-1}$ are deviations from the equilibrium relations, thus **VEqCM**.

If there is integration but no cointegration $\alpha\beta'$ is zero.

$$\Delta X_t = \mu + \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1} \Delta X_{t-k+1} + \epsilon_t \quad (2)$$

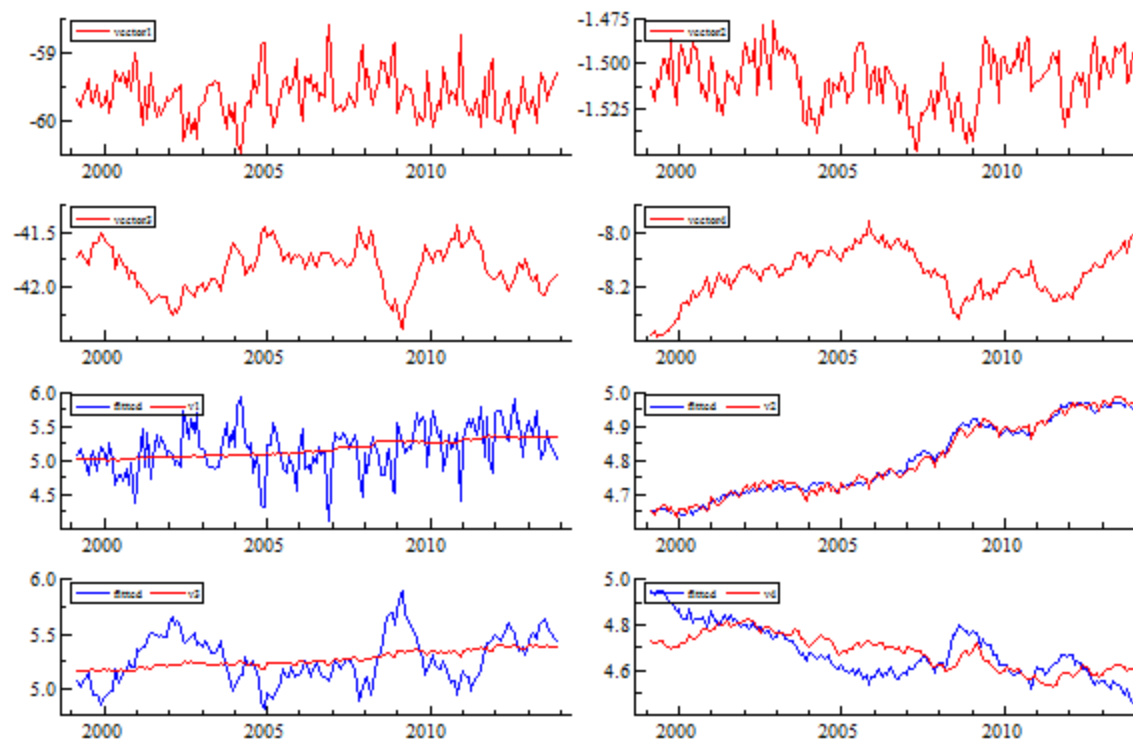
- μ : includes the marginal mean of the increments.
- $\alpha\beta' X_{t-1}$: incorporates the long run.
- The rest are **transitory dynamics**.

$$\Delta X_t = \mu + \alpha\beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \dots + \Gamma_{k-1} \Delta X_{t-k+1} + \epsilon_t \quad (2)$$

- μ : includes the marginal mean of the increments.
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- The rest are **transitory dynamics**.
- $\Pi = -(I - \Phi_1 \dots - \Phi_p)$
- **Useful plots:**
 - the cointegration relationships: disequilibrium component.
 - the long run: $X_t = \mu - (-\Phi_1 \dots - \Phi_p) X_{t-1}$.

EXAMPLE

- Four series:
- Prices of rice, canned fruit, spices and real exchange rate in US.
- Last four plots: original data with the long terms components.
- The first four plots: possible cointegration restrictions. After testing, only the first two are significant.



- **10 may**

Weak exogeneity and Granger causality in cointegrated systems

- In a bivariate system, like the one in the next slide, one variable (X_{2t}) is weakly exogenous if the adjustment coefficient in its equation (α_2) is zero. (Otherwise the coefficients of both equation would not be variation free parameters)
- Now X_{1t} does not Granger caused X_{2t} if X_{2t} does not respond to:
 - the lags of X_{1t} and
 - to the cointegration relationship.

AN EXAMPLE OF A COINTEGRATED WITHOUT TRANSITORY DYNAMICS, VAR(1) FOR n=2.

- The general formulation of the VAR

- $$\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [1]$$

•

- The general formulation in terms of matrix Π

- $$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 & \phi_{12} \\ \phi_{21} & \phi_{22} - 1 \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [2]$$

•

- A formulation with a matrix Π with r=1

- $$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 & \phi_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [3]$$

AN EXAMPLE OF A COINTEGRATED VAR(1) FOR n=2 / cont..

- The formulation of model (3) in terms of α and β

$$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\phi_{12}}{\phi_{11} - 1} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [4]$$

- The VEqCM form of model (4)

$$\begin{aligned} \Delta X_{1t} &= a_{11}(X_{1t-1} - \beta_{12}X_{2t-1}) + a_{1t} \\ \Delta X_{2t} &= a_{2t} \end{aligned} \quad [5]$$

- α_{11} (*adjustment coefficient*) = $(\phi_{11} - 1)$
- β_{12} (*long - run elasticity*) = $\phi_{12} / (1 - \phi_{11})$
- $\alpha_{12} = 0$ is the weak exogeneity condition.

GENERALIZATION OF THE BIVARIATE CVAR FOR MORE LAGS AND MORE VARIABLES

- $\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\phi_{12}}{\phi_{11}-1} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix}$
- $+ \textit{transitory dynamics} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$
- GENERALIZATION
- 1. Full alfa vector
- 2. Transitory dynamics.
- 3. More variables: α and β become matrices.

PROPERTIES OF THE CVAR

(Juselius 2014)

- Solve the problems of serial correlation in the residuals.
- Avoid spurious regression results.
- Eliminates multicollinearity.
- Gets normality. (It requires adjustments for previous breaks in the conditional means).
- Solve the problem of reduced rank dimension of the system.

- **COINTEGRATED
SYSTEMS WITH
DETERMINISTIC
ELEMENTS**

(See Enders 204).

I(1,1) COINTEGRATED SERIES.

- If the series are $I(1,1)$ they contain deterministic trends.
- The trend in there eventual forecasting function is linear with:
- stochastic intercept, depending on the initial conditions, and
- a deterministic slope.

The deterministic elements in a CVAR

- They can appear in:
 - the cointegration relationships and
 - in the equations of the system.

Dealing with variables $I(1,1^s)$ the deterministic elements would be **at most first order time polynomials.**

NOTES

- 1.- A linear polynomial in a $I(1,1)$ variable, X_{jt} , could appear in the cointegration relationships but not in the equation for ΔX_{jt} . If it did the variable would be affected by a quadratic trend.
- 2.- If a variable is $I(1,1)$ in the equation must appear a constant term.

.

$I(1,1)$:

$$(1) y_t = \tau t + y_t^* \quad (1) y_t^* = I(1,0)$$

$$(2) x_t = \rho t + x_t^* \quad (2) x_t^* = I(1,0)$$

If they are cointegrated:

$$y_t = c + \beta x_t + N_t \quad (3)$$

The unit root disappears in

$$N_t = y_t - c - \beta x_t, \quad (4)$$

But we need to see what happens with the deterministic trend..

Substitute in (3) $y_t = y + x_t$ for its values in (1) y (2). Thus

$$N_t = [\tau t - c - \beta \rho t] + [y_t^* - \beta x_t^*] . \quad (5)$$

If $y_t = y + x_t$ are cointegrated,
the second term in the RHS of (5) is stationary and
the first a linear deterministic trend,
unless that β also cancels the deterministic

This last point seems more important.

Consequently, when series are $I(1,0)$ we could test cointegration in the equation

$$y_t = c + \beta x_t + r_t$$

But if the series are $I(1,1)$ the model for testing cointegration must include a deterministic trend.

$$y_t = c + \gamma t + \beta x_t + r_t$$

$$y_t = c + \gamma t + \beta x_t + r_t \quad (5.a)$$

Under cointegration r_t is $I(0)$. **Two cases:**

(1) $\gamma \neq 0$, the cointegration relationship includes a linear trend, then the variables y_t and x_t diverge along time. The trend in **the eventual forecasting functions** in both cases have two elements. Intercept (depending on the initial conditions) and unrelated deterministic slopes between variables.

From (5.a) we could say that the variables are **cointegrated after detrending**.

(2) $\gamma = 0$, then the long run relationship $y_t - c - \beta x_t$ is stationary.

Of course the trend in **the eventual forecasting functions** continues to have two components –this is a property of the data which does not change with cointegration- but with drifts related by the long run elasticity.

In (2) we have cointegration which also cancels the deterministic trends (drifts).

In (1) cointegration without canceling the deterministic trends.

Two cases when cr cancels the drifts. γ is zero in the cr.

- $\tau = \beta \times \rho$
- β different than one

the series have different drift and differ in the long run.

$$\beta = 1,$$

both series have the same drift.

Alternative models to $I(1,1)$

- $I(2,0)$ the cointegration analysis is much more complex
- $I(1, 1^s)$ in which case the study of **cobreaking** is of great interest.

This last case can be very real in comparing prices at regional and national level.

The drifts and the constant terms in the cr's are not identified.

- Consider the following system [A] for two I(1,1) variables:
- $\Delta y_t = a_1 + \alpha_1 m_{t-1} + e_{1t}$
- $\Delta z_t = a_2 + \alpha_2 m_{t-1} + e_{2t}$,
- m_{t-1} is the cr.
- This can always be written as system [B]:
- $\Delta y_t = (a_1 - \alpha_1 c) + \alpha_1 (m_{t-1} + c) + e_{1t}$
- $\Delta z_t = (a_2 - \alpha_2 c) + \alpha_2 (m_{t-1} + c) + e_{2t}$.
- The transformation in [B] alters the drifts and the mean of the cr, but the resulting expression is also stationary and therefore a valid representation of the cr. This differs from regression model.
- **CONCLUSION:** The drifts and the intercept in the cr are not identified.

The equilibrium rates of growth

- $\Delta y_t = a_1 + \alpha_1 m_{t-1} + e_{1t}$
- $\Delta z_t = a_2 + \alpha_2 m_{t-1} + e_{2t}$,
- The equilibrium rates of growth of y_t and z_t are:
- $\mu_y = a_1 - \alpha_1 m^*$ and
- $\mu_z = a_2 - \alpha_2 m^*$, where m^* is the mean of the cr.
- Therefore, a useful way to solve the mentioned identification problem is by setting $m^*=0$.
- In this case the constants in the CVAR give us directly the equilibrium rates of growth of the variables.
- Note that the drift parameters in the CVAR are not identified but the equilibrium rates of growth are.

COINTEGRATED I(1,1) SERIES

- Being I(1,1) **they have a linear trend.**
- The equations in the CVAR **must have a constant.**
- **The equilibrium rate of growth is identified but the drifts are not.** The identification with a zero mean in the cr is useful.
- The cointegration relationship could cancel the deterministic trends and they do not appear in the cr.
- In the above case **the value of β is important.** β different than one implies that both series diverge in the long run by a deterministic trend.
- **If the stochastic cointegration does not cancel the drifts, the cointegration relationship must include a trend.**
- **All this must be tested.**

Plots from Enders p 350.

- A) series are $I(1,0)$ and cointegrated, with no constant in the cr. No deterministic trend.
- B) and C) They are $I(1,1)$ and the cr cancels the unit roots and the deterministic trends.
- D) they are $I(1,0)$, but the cr has a constant.

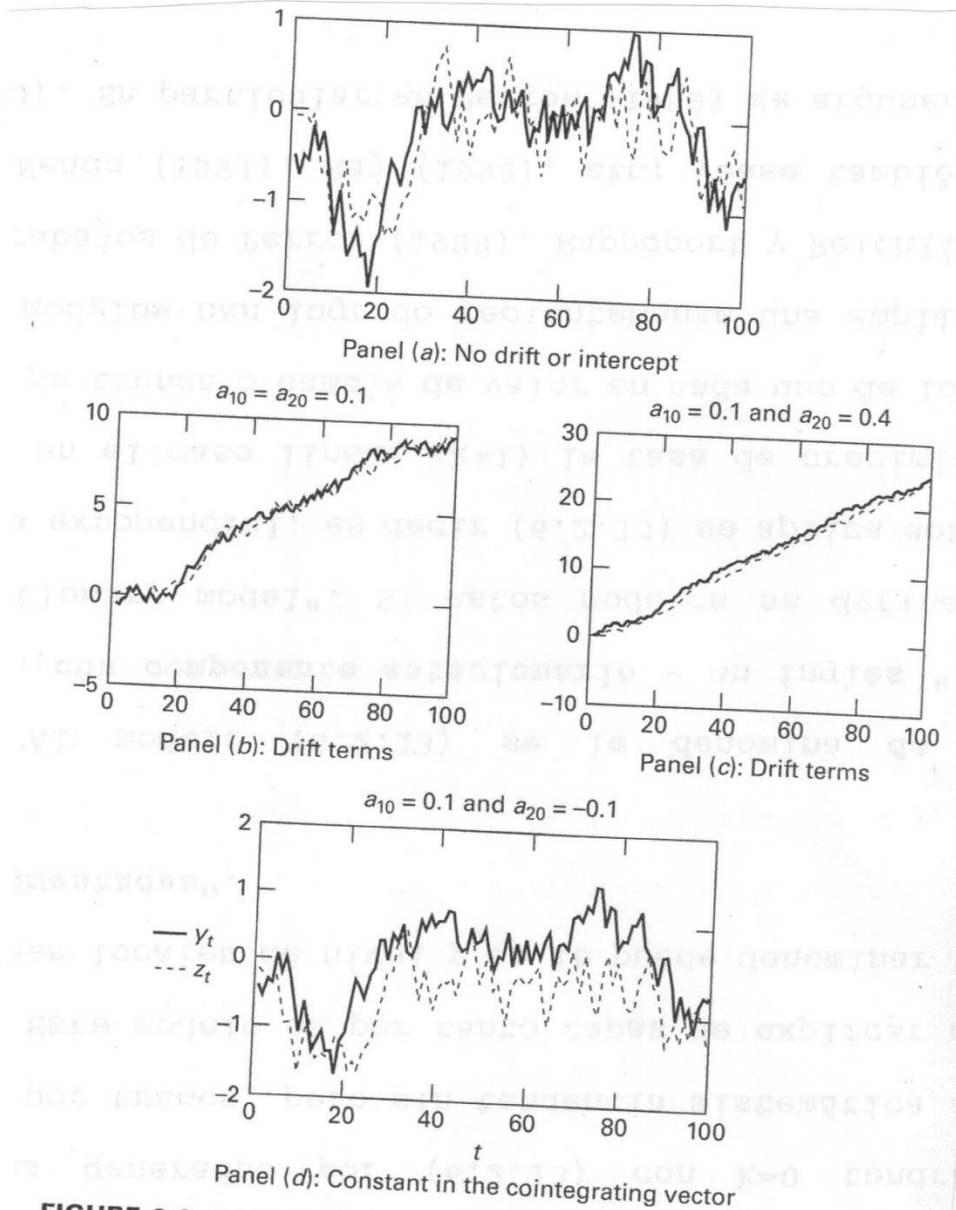


FIGURE 6.3 Drifts and Intercepts in Cointegrating Relationships

COINTEGRATION IN ADL MODELS.

See Hendry and Nielsen pg 267

EqCM REPRESENTATION OF THE GENERAL ADL MODEL

Let us consider the general ADL model with only one explanatory variable and without intercept given by the following equation

$$a(L)y_t = b(L)x_t + \varepsilon_t \quad (A.1)$$

where $a(L) = (1 - \sum_{j=1}^r a_j L^j)$, $b(L) = (\sum_{j=0}^s b_j L^j)$ and ε_t is white noise. The long run relationship between y_t and x_t is given by

$$y = \frac{b(1)}{a(1)} x = \frac{b_0 + b_1 + \cdots + b_s}{1 - a_1 - \cdots - a_r} x.$$

where the coefficient $b(1)/a(1)$ is the long-run gain in y_t with respect to x_t .

To derive the ECM representaci3n of (A.1) we will make use of a known result which states that any polynomial

$\delta(L) = \delta_0 + \delta_1 L + \cdots + \delta_p L^p$ can be descomposed as

$$\delta(L) = \delta + \delta^*(L)(1 - L) \tag{A.2}$$

where $\delta = \delta(1)$ and $\delta^*(L)$ is a polynomial of order $p-1$ whose coefficients are obtained by equating powers of L in both sides of equation (A.2).

Let us write the polynomial $a(L)$ in (A.1) as

$$a(L) = 1 - L\bar{a}(L) \tag{A.3}$$

where $\bar{a}(L) = a_1 + a_2L + \cdots + a_{r-1}L^{r-1}$. Applying formula (A.2) to this polynomial $\bar{a}(L)$ and substituting in (A.3), $a(L)$ can be alternatively written as:

$$\mathbf{a}(L) = \mathbf{1} - \bar{\mathbf{a}}L - L\bar{\mathbf{a}}^*(L)(\mathbf{1} - L). \tag{A.4}$$

where $\bar{a} = \bar{a}(1)$.

Applying now formula (A.2) to the polynomial $b(L)$ it can be written as

$$\mathbf{b}(L) = \mathbf{b} + \mathbf{b}^*(L)(\mathbf{1} - L). \quad (A.5)$$

Where $b = b(1)$. Putting back expressions (A.4) and (A.5) into equation (A.1) yields

$$\mathbf{y}_t = \bar{\mathbf{a}}\mathbf{y}_{t-1} + \bar{\mathbf{a}}^*(L)\Delta\mathbf{y}_{t-1} + \mathbf{b}\mathbf{x}_t + \mathbf{b}^*(L)\Delta\mathbf{x}_t + \boldsymbol{\varepsilon}_t.$$

If y_{t-1} is subtracted from both sides of (A.1) and bx_{t-1} is subtracted and added on the right hand sides of it, then the above equation becomes

$$\Delta y_t = (\bar{a} - 1)y_{t-1} + \bar{a}^*(L)\Delta y_{t-1} + [b + b^*(L)]\Delta x_t + bx_{t-1} + \varepsilon_t,$$

and this can be alternative written as

$$\Delta y_t = \bar{a}^*(L)\Delta y_{t-1} + b^{**}(L)\Delta x_t + \alpha(y_{t-1} - \beta x_{t-1}) + \varepsilon_t, \quad (A6)$$

where $\alpha = (\bar{a} - 1)$, $\beta = \frac{b}{1-\bar{a}}$ and $b^{**}(L) = b + b^*(L)$. However,

from (A.3) it can be immediately seen that $(1 - \bar{a}) = a$, so that

$\beta = b/a = b(1)/a(1)$, i.e. the long-run gain in y_t with respect to x_t .

Therefore, equation (A.6) is the EqCM formulation of model (A.1).

- **This formulation explains changes in the current endogenous variable in terms of**
- changes in its own past and the past in the explanatory variable, plus
- an adjustment to the past equilibrium error, gathered in the term $\alpha(y_{t-1} - \beta x_{t-1})$, and
- a white noise disturbance ε_t .
- Thus, this model is a stationary formulation for a relationship between non-stationary variables, making use of the restriction that ties them in the long run as they are cointegrated.

AN EXAMPLE OF A COINTEGRATED WITHOUT TRANSITORY DYNAMICS, VAR(1) FOR n=2.

- The general formulation of the VAR

- $$\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [1]$$

-

- The general formulation in terms of matrix Π

- $$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 & \phi_{12} \\ \phi_{21} & \phi_{22} - 1 \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [2]$$

-

- A formulation with a matrix Π with r=1

- $$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 & \phi_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [3]$$

AN EXAMPLE OF A COINTEGRATED VAR(1) FOR n=2 / cont..

- The formulation of model (3) in terms of α and β

$$\begin{pmatrix} \Delta X_{1t} \\ \Delta X_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} - 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\phi_{12}}{\phi_{11} - 1} \end{pmatrix} \begin{pmatrix} X_{1t-1} \\ X_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \quad [4]$$

- The VEqCM form of model (4)

$$\begin{aligned} \Delta X_{1t} &= a_{11}(X_{1t-1} - \beta_{12}X_{2t-1}) + a_{1t} \\ \Delta X_{2t} &= a_{2t} \end{aligned} \quad [5]$$

- α_{11} (*adjustment coefficient*) = $(\phi_{11} - 1)$
- β_{12} (*long - run elasticity*) = $\phi_{12} / (1 - \phi_{11})$
- $\alpha_{12} = 0$ is the weak exogeneity condition.

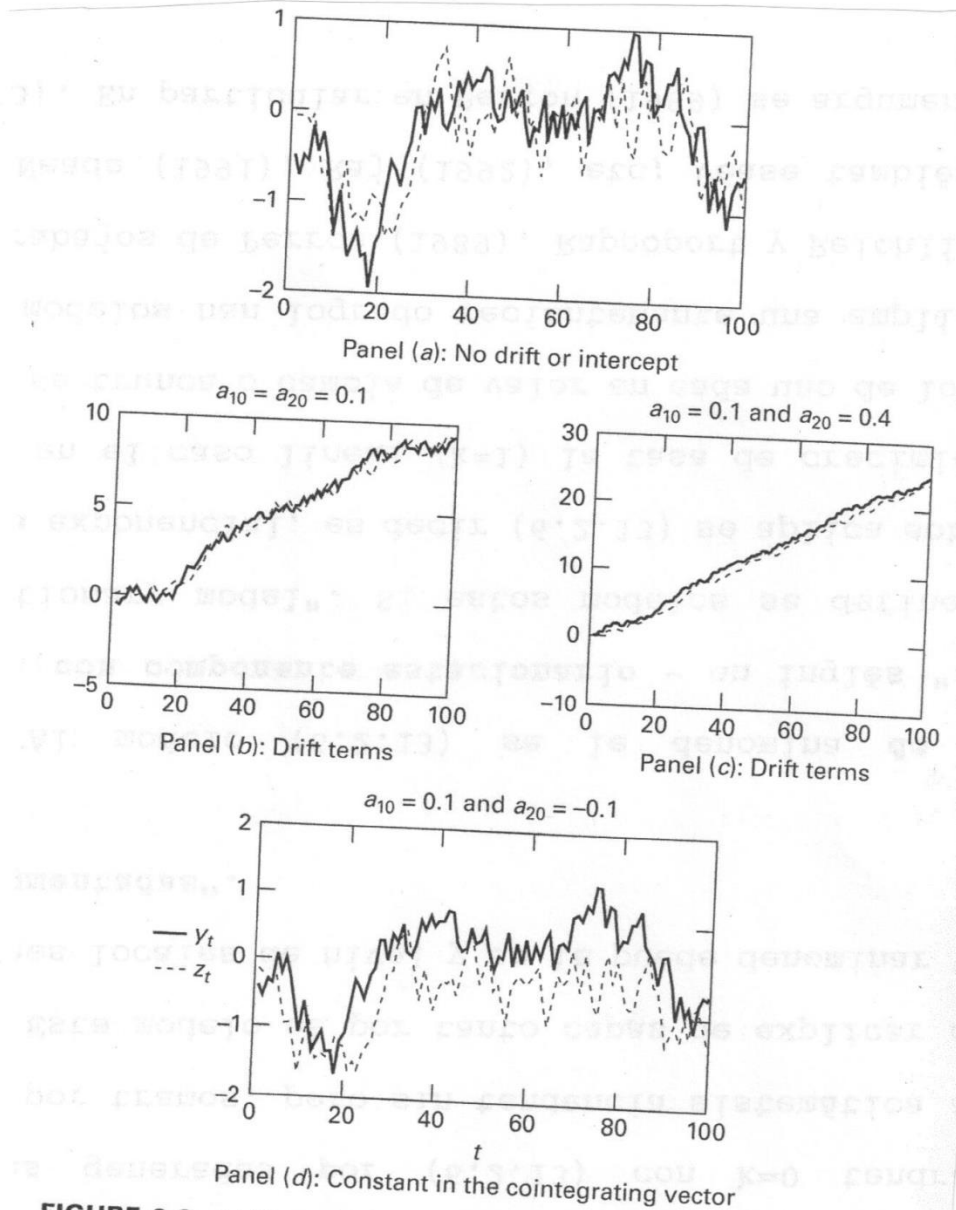


FIGURE 6.3 Drifts and Intercepts in Cointegrating Relationships

*Dynamic relationship between two
international prices of meat.*

Hernandez et al. (2002)

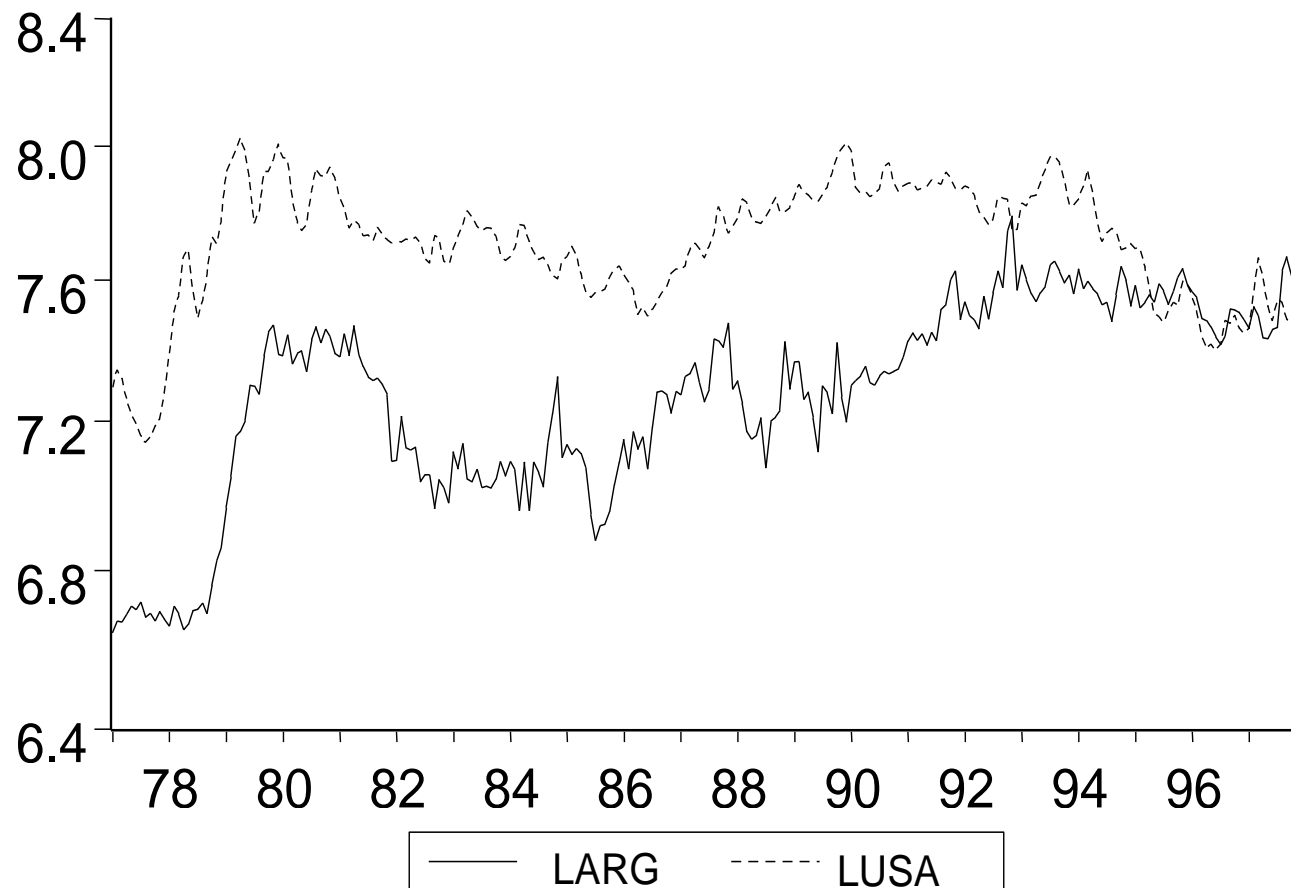
The aim is to study by cointegration analysis if there is integration in the international market for beef.

The aphtose fever in Argentina segmented the international market of beef in two.

The export prices of Argentina beef and the import prices in the US for Australian beef are representative of the prices in two sectors.

Monthly data between 1997/1 and 1997/12.

Gráfico 1.- Logaritmos de los precios mensuales de carne de vacuno en Estados Unidos (LUSA) y Argentina (LARG) durante el periodo 1977-1997



ADF tests

	Valor del Estadístico	Valores Críticos (1%, 5%, 10%)
LUSA (4)	-2,833254	-3,9984, -3,4292, -3,1378
DLUSA (4)	-7,572660	-3,4585, -2,8734, -2,5730
LARG (4)	-2,610202	-3,9984, -3,4292, -3,1378
DLARG (4)	-7,482174	-3,4585, -2,8734, -2,5730

BREAKS IN THE DATA

- Till the end of 1990 both prices follow a similar trend.
- That changes in 1991.
- From 1994 they again show a similar trend.

TWO DUMMY VARIABLES: $D1_t$ y $D2_t$.

- $D1_t = 0$ till December 1990, then

the values 1,2,3,.. till december 1994 and

zero values afterwards.

La variable $D2t$ is
a step dummy
with zero values till 1994 and
value 1 from January 1995.

$$\begin{aligned}
\Delta LARG_t = & - \frac{0,13}{(0,03)} * \left(\frac{0,51}{(0,03)} + LARG_{t-1} - LUSA_{t-1} - \frac{0,008}{(0,002)} D_{1t-1} - \frac{0,52}{(0,11)} D_{2t-1} \right) \\
& - \frac{0,19}{(0,06)} * \Delta LARG_{t-1} + \frac{0,01}{(0,06)} * \Delta LARG_{t-2} + \frac{0,18}{(0,06)} * \Delta LARG_{t-3} - \frac{0,09}{(0,06)} * \Delta LARG_{t-4} \\
& + \frac{0,02}{(0,11)} * \Delta LUSA_{t-1} + \frac{0,05}{(0,12)} * \Delta LUSA_{t-2} - \frac{0,15}{(0,11)} * \Delta LUSA_{t-3} - \frac{0,16}{(0,14)} * \Delta LUSA_{t-4}
\end{aligned}$$

$$\begin{aligned}
\Delta LUSA_t = & \frac{0,008}{(0,01)} * \left(\frac{0,51}{(0,03)} + LARG_{t-1} - LUSA_{t-1} - \frac{0,008}{(0,002)} D_{1t-1} - \frac{0,52}{(0,11)} D_{2t-1} \right) \\
& - \frac{0,0004}{(0,04)} * \Delta LARG_{t-1} + \frac{0,09}{(0,04)} * \Delta LARG_{t-2} + \frac{0,03}{(0,04)} * \Delta LARG_{t-3} - \frac{0,06}{(0,04)} * \Delta LARG_{t-4} \\
& + \frac{0,46}{(0,06)} * \Delta LUSA_{t-1} - \frac{0,22}{(0,05)} * \Delta LUSA_{t-2} - \frac{0,04}{(0,07)} * \Delta LUSA_{t-3} - \frac{0,06}{(0,06)} * \Delta LUSA_{t-4}
\end{aligned}$$

Testing for residual correlation

- The white noise hypothesis is not rejected.

TRANSITORY DYNAMICS

- Seems restricted to the own lags, except perhaps a second lag effect of LARG in LUSA.
- In the long run LUSA is the leader, since do not depend on deviations from equilibrium.

Long run relationship:

$$\text{LARG}_t = \text{LUSA}_t - \underset{(0,002)}{0,008} * D_{1t} - \underset{(0,11)}{0,52} * D_{2t} - \underset{(0,03)}{0,51}$$

From 1995: LARG=LUSA

EXAMPLES OF US CONSUMER PRICES

10th May 2016. Topic 6

- Testing for cointegration Enders pg 335

Regression on I(1) independent variables.

Asymptotically a test on the regression coefficient would not reject the hypothesis that is zero.

On finite samples this hypothesis is often rejected by the standard procedures.

Yule () and Granger and Newbold ().

Making the regression in levels the residual term is non-stationary.

Spurious regressions

One gets high R^2 and high t statistics by standard procedures, but they are invalid since the errors are non-stationary.

You detect that the results are spurious analysing the residuals of the regression.

In relating $I(1)$ variables it is important to check for cointegration.

Example of spurious regression

- Regression between the CPI for bread and the investment in equipment.

Regresión Espuria

Precio (CPI):

Consumer Price Index –
All Urban Consumers

Series

Id: CUUR0000SEFB01

Not Seasonally Adjusted

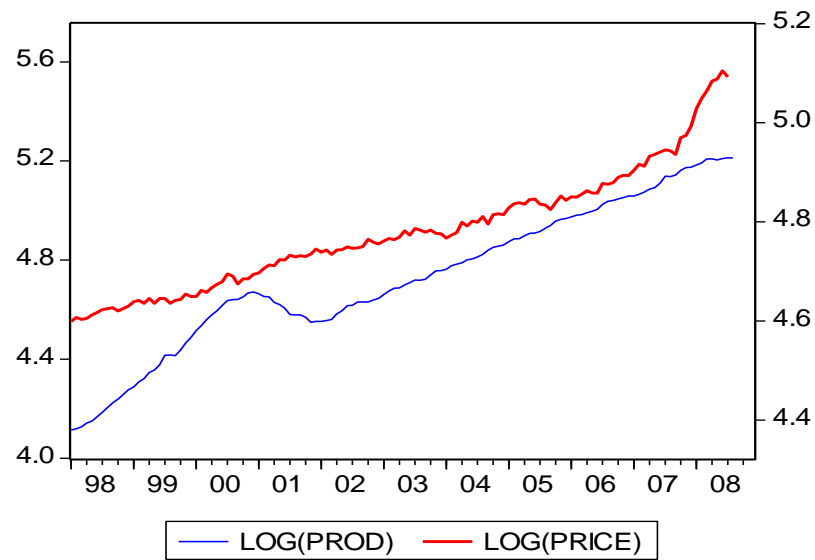
Area: U.S. city average

Item: Bread

Producción:

Equipment parts NACE CODE :B53120:

Gráfico 1



Regresión

Dependent Variable: LOG(PRICE)

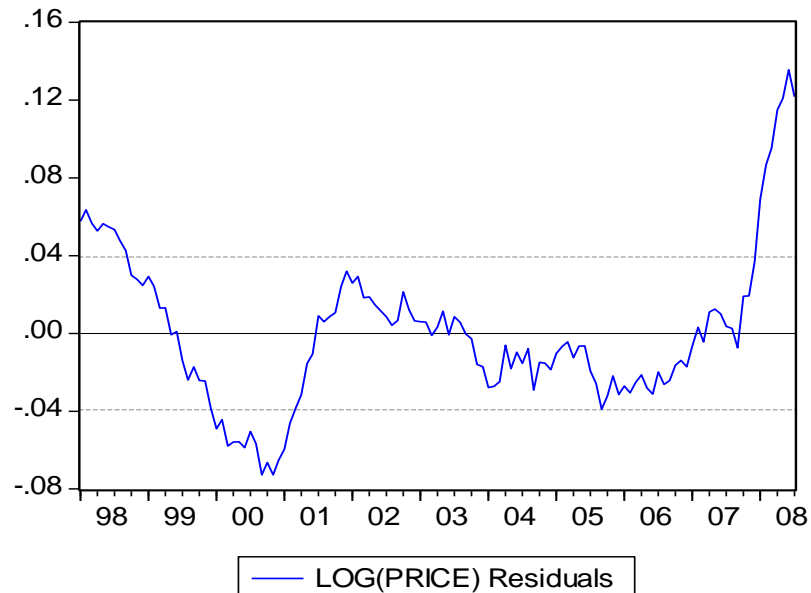
Method: Least Squares

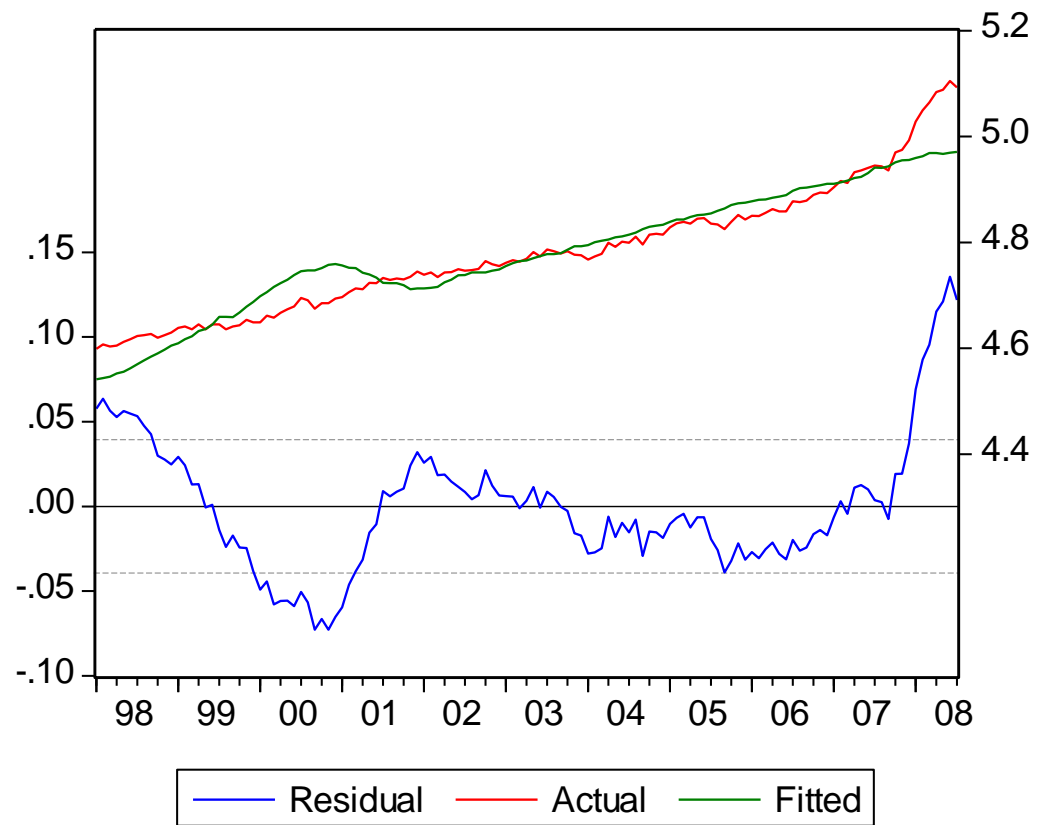
Date: 10/07/08 Time: 19:22

Sample (adjusted): 1998M01 2008M07

Included observations: 127 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	2.935175	0.057064	51.43623	0.0000
LOG(PROD)	0.390484	0.012056	32.38839	0.0000
R-squared	0.893527	Mean dependent var	4.779941	
Adjusted R-squared	0.892675	S.D. dependent var	0.119988	
S.E. of regression	0.039309	Akaike info criterion	-3.619117	
Sum squared resid	0.193147	Schwarz criterion	-3.574326	
Log likelihood	231.8139	F-statistic	1049.008	
Durbin-Watson stat	0.057374	Prob(F-statistic)	0.000000	





Unit root test

Null Hypothesis: RESID11 has a unit root

Exogenous: Constant

Lag Length: 3 (Automatic based on AIC, MAXLAG=12)

	t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic	-1.641662	0.4583
Test critical values: 1% level	-3.484198	
5% level	-2.885051	
10% level	-2.579386	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation

Dependent Variable: D(RESID11)

Method: Least Squares

Date: 10/07/08 Time: 19:24

Sample (adjusted): 1998M05 2008M07

Included observations: 123 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
RESID11(-1)	-0.040368	0.024589	-1.641662	0.1033
D(RESID11(-1))	-0.011872	0.092406	-0.128476	0.8980
D(RESID11(-2))	0.339812	0.089491	3.797154	0.0002
D(RESID11(-3))	0.258335	0.094109	2.745052	0.0070
C	0.000192	0.000808	0.237694	0.8125
R-squared	0.153219	Mean dependent var		0.000562
Adjusted R-squared	0.124515	S.D. dependent var		0.009472
S.E. of regression	0.008863	Akaike info criterion		-6.574136
Sum squared resid	0.009269	Schwarz criterion		-6.459820
Log likelihood	409.3094	F-statistic		5.337813
Durbin-Watson stat	2.006555	Prob(F-statistic)		0.000550

Regresión sobre las diferencias:

Dependent Variable: D(LOG(PRICE))

Method: Least Squares

Date: 10/08/08 Time: 15:55

Sample (adjusted): 1998M02 2008M07

Included observations: 126 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.003271	0.001051	3.113616	0.0023
D(LOG(PROD))	0.073866	0.079186	0.932814	0.3527
R-squared	0.006968	Mean dependent var		0.003916
Adjusted R-squared	-0.001040	S.D. dependent var		0.008880
S.E. of regression	0.008884	Akaike info criterion		-6.593339
Sum squared resid	0.009787	Schwarz criterion		-6.548319
Log likelihood	417.3804	F-statistic		0.870142
Durbin-Watson stat	2.187718	Prob(F-statistic)		0.352729

Types of long run relationships between $I(1)$ variables.

- 1.- If they are independent there is no long run relationship between them. Still a regression in levels could appear as very significant by traditional standards. This **regression is spurious**.
- 2.- The variables are not related in the long run but they are in the short run. Formulate a **model on the differences of the variables**.
- 3.- There is a long run restriction. Formulate a **Equilibrium correction model**.

Two ways of seeing and testing cointegration

- 1.- The **error term** of a cointegration relationship (a regression between $I(1)$ variables) must be **stationary**.

Run the regression and test for stationarity in the residuals. **Engle-Granger test**.

- 2.- The **rank of matrix Π** must have a restricted non-zero rank.

Test the rank of Π . **Johansen test**.

Engle-Granger method: two-step procedure.

(Enders sección 6,4 and 6.7)

- **Testing if two $I(1)$ variables are cointegrated..**
- 1.- Use the ADF test for integration..
- 2.- Then run a regression between the variables in levels.

(Step 1)

Test if the residuals are stationary.(step 2)

3.- If you don't reject the stationary hypothesis use directly the residuals to formulate an EqCM.

4.- Augmented E-G test. Use sufficient number of lags in step one.

ENGLE-GRANGER TEST: some limitations.

- - **(1)** Changing the dependent variable in the regression between two variables we can get in small samples different conclusions.
- - **(2)** The method can allow for more variables in the regression – increasing the problem in (1)- but cannot face the problem of more than one cointegration relationship.
- - **(3)** Errors from the regression in the first step are carried into the second step testing for a unit root.

Johansen procedure

- It is also a generalization of the Dickey-Fuller test.

Alternatives in the rank of Π

The general model (G) after reduction process

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \dots + \Gamma_{p-1} \Delta x_{t-p+1} + a_t$$

The term Πx_{t-1} captures the long term,
all the other elements are stationary, non-persistence.

$r = 0$ the model is a VAR on first differences.

(no restrictions in the long run)

$R = n$ the model is a VAR on levels. (Variables are stationary)

$0 < r < n$ the long run is restricted. A restricted VAR in Π .

This VAR can be formulated in terms of the
equilibrium correction mechanism.

$$\Pi X_t$$

- If Π has full rank, ΠX_t is the long run solution of a system of n equations.
- These equations are linear regressions for the n X_t variables. Thus if rank is n , all the n variables are restricted to be stationary.

Johansen procedure

- Enders 6.7
- **Johansen's procedure:**
- 1.- The test is run using **maximum likelihood** and thus avoids the inconvenient of the two-step method of Engle-Granger.
- 2.- Can estimate and test for **more than one cointegration restriction**.
- 3.- **Can test restricted versions** of the cointegration relationships.
- 4.- **Can test hypotheses on the speed of adjustment parameters**, and in particular hypotheses of weak exogeneity by putting zero restrictions.

Johansen procedure is nothing more than a multivariate generalization of the Dickey-Fuller test. In the univariate case, it is possible to view the stationarity of $\{y_t\}$ as being dependent on the magnitude $(a_1 - 1)$; that is,

$$y_t = a_1 y_{t-1} + \varepsilon_t$$

or

$$\Delta y_t = (a_1 - 1)y_{t-1} + \varepsilon_t$$

If $(a_1 - 1) = 0$, the $\{y_t\}$ process has a unit root. Ruling out the case in which $\{y_t\}$ is explosive, if $(a_1 - 1) \neq 0$ we can conclude that the $\{y_t\}$ sequence is stationary. The Dickey-Fuller tables provide the appropriate statistics to formally test the null hypothesis $(a_1 - 1) = 0$. Now consider the simple generalization to n variables; as in (6.26), let

$$x_t = A_1 x_{t-1} + \varepsilon_t$$

so that

$$\begin{aligned} \Delta x_t &= A_1 x_{t-1} - x_{t-1} + \varepsilon_t \\ &= (A_1 - I)x_{t-1} + \varepsilon_t \\ &= \pi x_{t-1} + \varepsilon_t \end{aligned} \tag{6.50}$$

where: x_t and ε_t are $(n \cdot 1)$ vectors

A_1 = an $(n \cdot n)$ matrix of parameters

I = an $(n \cdot n)$ identity matrix

π is defined to be $(A_1 - I)$

THE NUMBER OF COINTEGRATING RELATIONSHIPS IS GIVEN BY THE RANK OF Π

- ΠX_t gives the long run relationships between the variables.
- The rank of Π gives the number of independent long run relationships which are stationary.
- The rank of Π equals to the number of non-zero characteristic roots of matrix Π .
- The procedure starts testing the hypothesis of zero unit roots against one .

SOME QUESTIONS ON BOTH TESTS

- Being tests on unit roots their asymptotic distributions are unknown and must be simulated.
- In large samples these distributions are independent of the transitory dynamics, but in finite samples usually an adequate specification of the lags matters.
- The asymptotic distributions depend on the deterministic elements in the model.

COINTEGRATION AND OUTLIERS

- If the data have outliers and the model does not incorporate factors for their correction, the model is incorrect and usually biased the results.
- Uncorrected step outliers can lead to wrongly infer cointegration.
- Cointegration analysis requires outliers correction.
- With dummies for the outliers the asymptotic distribution is different.