Profit maximisation 1

Assume now

- L-dimensional vector of prices $\mathbf{p} = (p_1, ..., p_L) > 0$, independent from the choices of the firm:
  $\Rightarrow$ firm is **price taker** in input and output markets
- firm maximises profits
- Y is not empty, closed and satisfies free disposal

The firm's problem can be stated as

$$\begin{array}{ll} \max_{\mathbf{y}} & \mathbf{p} \cdot \mathbf{y} \\ \text{s.t} & \mathbf{y} \in Y \end{array} \qquad \text{PMP}$$

or, equivalently

$$\begin{array}{rl} \max\limits_{\mathbf{y}} & \mathbf{p} \cdot \mathbf{y} \\ \text{s.t} & F(\mathbf{y}) \leq 0 \end{array}$$

Profit maximisation 2

If $F(.)$ is differentiable, necessary condition for profit maximisation are

$$\mathbf{p} = \lambda^* \, \nabla F(\mathbf{y}^*) \qquad \lambda^* \geq 0 \qquad\qquad \text{FOC-PMP}$$

$\Rightarrow$ **y** is chosen so that **p** and $\nabla F(\mathbf{y}^*)$ are *proportional* $\Leftarrow$

If Y is convex, FOC-PMP is not only necessary but also sufficient for profit maximisation.

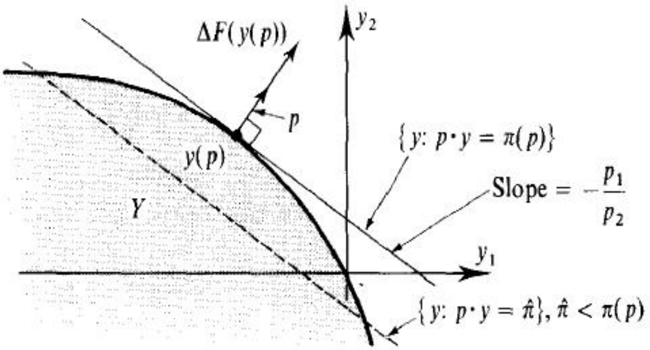
More than simply a technical point!!

For instance, Assume $L = 2$ and good 1 being a net output and 2 a net input. If Y shows CRS or IRS, then $y_1^* = \infty$ when p_1 sufficiently large relatively to p_2, and $y_1^* = 0$ otherwise.

Profit maximisation 3

FOC-PMP can be rewritten as follows, for any $k, l = 1, .., L$ and $k \neq l$:

$$\frac{p_k}{p_l} = \frac{\frac{\partial F(\mathbf{y}^*)}{\partial y_k}}{\frac{\partial F(\mathbf{y}^*)}{\partial y_l}} = MRT_{kl} \qquad \text{FOC-PMP2}$$



Profit function and supply correspondence

Two fundamental functions/correspondences ONLY deriving from the profit maximising behaviour hypothesis are:

- the **profit function**

$$\pi(\mathbf{p}) = \mathbf{p} \cdot \mathbf{y}^*$$

  which associates to every $\mathbf{p}$ the maximum value of $\mathbf{p} \cdot \mathbf{y}$;

- the **supply correspondence**

$$\mathbf{y}(\mathbf{p}) = \{\mathbf{y} \in Y : \mathbf{p} \cdot \mathbf{y} = \pi(\mathbf{p})\}$$

  which associates to every $\mathbf{p}$ the profit maximising production plan $\mathbf{y}^*$.

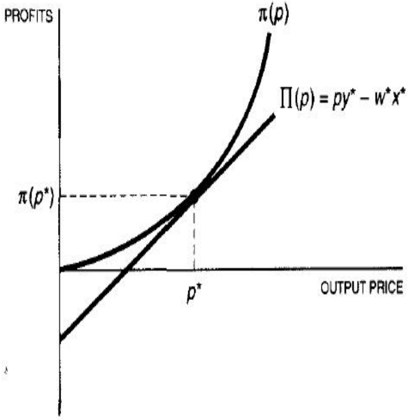
Properties of the profit and supply functions/correspondences

- If Y is convex, $\mathbf{y}(\mathbf{p})$ is a convex set for all $\mathbf{p}$. If Y is strictly convex, $\mathbf{y}(\mathbf{p})$ is single-valued.
- If Y is convex, then
  $Y = \{\mathbf{y} \in \Re^L : \mathbf{p} \cdot \mathbf{y} \leq \pi(\mathbf{p}) \text{ for all } \mathbf{p} \gg 0\}$.

  The profit function is a complete description of the technology.

Properties of the profit and supply functions/correspondences 2

- $\pi(.)$ is **convex in prices**.
  Let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ for all $0 \le t \le 1$. Then,
  $\pi(\mathbf{p}'') \le t\pi(\mathbf{p}) + (1-t)\pi(\mathbf{p}')$.



Properties of the profit and supply functions/correspondences 3

- When $\pi(.)$ is differentiable, can obtain the supply correspondence from the profit function, using the **Hotelling's lemma**

$$\nabla \pi(\mathbf{p}) = \mathbf{y}(\mathbf{p})$$

or, equivalently,

$$\frac{\partial \pi(\mathbf{p})}{\partial p_i} = y_i(\mathbf{p}) \quad \text{for } i = 1, .., L.$$

(when i is an input, $y_i(\mathbf{p})$ is usually referred to as **factor demand function**).

Hotelling's lemma is simply an application of the envelope theorem (see previous picture).

Properties of the profit and supply functions/correspondences 4

- $D\mathbf{y}(\mathbf{p})$ is positive semidefinite.

  Because of Hotelling's lemma, $D\mathbf{y}(\mathbf{p}) = D^2\pi(\mathbf{p})$. Since $\pi(.)$ is convex, its Hessian matrix must be positive semidefinite, so that also $D\mathbf{y}(\mathbf{p})$ must be positive semidefinite.

Positive semidefiniteness of $D\mathbf{y}(\mathbf{p})$ implies...

Properties of the profit and supply functions/correspondences 5

1. the principal-minor determinants are all positive.
   $\hookrightarrow$ technical requirement for convexity of the supply function
2. $D\mathbf{y}(\mathbf{p})$ is symmetric: cross-substitution effects are symmetric

$$\frac{\partial^2 \pi(.)}{\partial p_\ell \partial p_k} = \frac{\partial y_\ell(.)}{\partial p_k} = \frac{\partial y_k(.)}{\partial p_\ell} = \frac{\partial^2 \pi(.)}{\partial p_k \partial p_\ell}. \qquad \text{for } \ell, k = 1.., L$$

   $\hookrightarrow$ very little intuition...
3. **law of supply**: own-price effects are nonnegative

$$\frac{\partial y_\ell(.)}{\partial p_\ell} \geq 0 \ \text{ for } \ell = 1.., L.$$

   $\hookrightarrow$ optimal amount of output increases as the price with its price and optimal amount of input decreases with its price

Properties of the profit and supply functions/correspondences 6

- $\pi(.)$ is **homogenous of degree one**
  $\mathbf{y}(\mathbf{p})$ is **homogenous of degree zero**
  For all $t > 0, \pi(t\mathbf{p}) = t\, \pi(\mathbf{p})$ and $\mathbf{y}(t\mathbf{p}) = \mathbf{y}(\mathbf{p})$.

  A proportional change of all prices change (optimal) profits by the same proportion but does not change the (optimal) production plan.

  The relationship between these two results follows from Hotelling's lemma, being the factor demands the derivative of the profit function.

Cost minimisation

A choice of inputs that minimises the cost of producing a given output is a necessary (but not sufficient) condition for profit maximisation.

Result on costs of interest because

- often more useful than results on technology, esp. in applied work
- require only price-taking assumption in input markets
- better accommodate constant or nondecreasing returns to scale

Focus on *single-output* technology (restrictive assumption).

Cost minimisation problem

To minimise costs, a firm solves the problem

$$\begin{array}{ccc} \min\limits_{\mathbf{z}} & \mathbf{w}\cdot\mathbf{z} & \qquad\qquad \text{CMP} \\ \text{s.t} & q \leq f(\mathbf{z}) & \end{array}$$

Necessary condition for $\mathbf{z}(q, \mathbf{w})$ to be the solution to CMP are, for some $\lambda \geq 0$ and for $\ell = 1, .., L-1$,

$$w_\ell \geq \lambda \frac{\partial f(\mathbf{z}^*)}{\partial z_\ell} \ \text{(with = when } z_\ell^* > 0) \ \text{FOC-CMP}$$

or, equivalently,

$$\mathbf{w} \geq \lambda \nabla f(\mathbf{z}^*) \ \text{ and } \ [\mathbf{w} - \lambda \nabla f(\mathbf{z}^*)] \cdot \mathbf{z}^* = 0$$

If $f(.)$ is quasi-concave, these conditions are also sufficient for cost minimization.

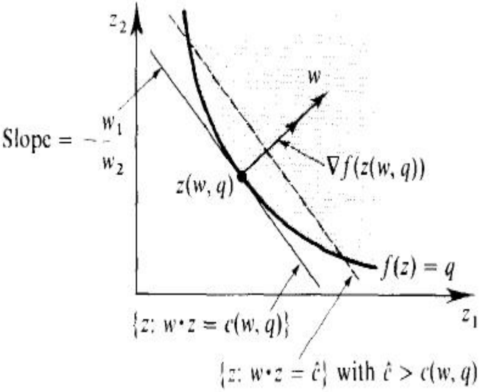
Cost minimisation problem 2

In case of interior solutions, FOC-CMP can be re-written as follows, for $\ell, k = 1, .., L$ and $\ell \neq k$,

$$\frac{w_\ell}{w_k} = \frac{\frac{\partial f(z_\ell^*, z_k^*)}{\partial z_\ell}}{\frac{\partial f(z_\ell^*, z_k^*)}{\partial z_k}} = MTRS_{\ell k} \qquad \text{(FOC-CMP2)}$$

which is clearly a special case of the condition FOC-PMP2 for profit maximisation and which has a nice graphical interpretation.

FOC-CMP



z_2
w
Slope $= -\frac{w_1}{w_2}$
$\nabla f(z(w, q))$
$z(w, q)$
$f(z) = q$
z_1
$\{z : w \cdot z = c(w, q)\}$
$\{z : w \cdot z = \hat{c}\}$ with $\hat{c} > c(w, q)$

From CMP

Two fundamental functions/correspondences deriving from cost minimisation problem:

- the **conditional factor demand correspondence**

$$\mathbf{z}(q, \mathbf{w})$$

  which associates to every q and $\mathbf{w}$ the cost minimising input demand

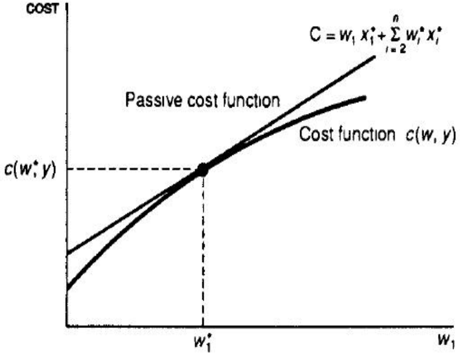
- the **cost function**

$$c(q, \mathbf{w}) = \mathbf{w} \cdot \mathbf{z}(q, \mathbf{w})$$

  which associates to every q and $\mathbf{w}$ the minimum production cost

Properties of cost fct and conditional demand factor fct 1

- $c(.)$ is concave in $\mathbf{w}$.



Properties of cost fct and conditional demand factor fct 2

- If the sets $\{\mathbf{z} > 0 : f(\mathbf{z}) \geq q\}$ are convex for every q, then $Y = \{(-\mathbf{z}, q) : \mathbf{w} \cdot \mathbf{z} \geq c(\mathbf{w}, q) \text{ for all } \mathbf{w} > 0\}$.
  The cost function is a complete description of the technology.

Properties of cost fct and conditional demand factor fct 3

- When $c(.)$ is differentiable, can obtain the conditional factor demand correspondence from the cost function, using

$$\nabla_w c(\mathbf{w}, q) = \mathbf{z}(\mathbf{w}, q) \qquad \text{Shepard's lemma}$$

or, equivalently,

$$\frac{\partial c(\mathbf{w}, q)}{\partial w_i} = z_i^*(\mathbf{w}, q) \quad \text{for } i = 1, .., L.$$

Similarly to Hotelling's lemma, Shepard's lemma is simply an application of the envelope theorem (see previous picture).

Properties of cost fct and conditional demand factor fct 4

- $D_{\mathbf{w}}\mathbf{z}(\mathbf{w}, q)$ is symmetric negative semidefinite.
  Because of Shepard's lemma,$D_{\mathbf{w}}\mathbf{z}(\mathbf{w}, q) = D^2 c(\mathbf{w}, q)$.

  Since $c(.)$ is concave in $\mathbf{w}$, its Hessiam matrix must negative semidefinite, so that also $D_{\mathbf{w}}\mathbf{z}(\mathbf{w}, q)$ must be negative semidefinite.

  Negative semidefiniteness of $D_{\mathbf{w}}\mathbf{z}(\mathbf{w}, q)$ implies...

Properties of cost fct and conditional demand factor fct 4

1. the principal-minor determinants have alternate sign, starting from negative.
   $\hookrightarrow$ technical requirement for concavity of the cost function
2. $D_{\mathbf{w}}\mathbf{z}(\mathbf{w}, q)$ is symmetric:

$$\frac{\partial^2 c(.)}{\partial w_\ell \partial w_k} = \frac{\partial z_\ell(.)}{\partial w_k} = \frac{\partial z_k(.)}{\partial w_\ell} = \frac{\partial^2 c(.)}{\partial w_k \partial w_\ell}.$$

   $\hookrightarrow$ very little intuition...
3. the conditional factor demand are (weakly) downward sloping:

$$\frac{\partial z_i(.)}{\partial w_i} = \frac{\partial^2 c(.)}{\partial w_i} \leq 0 \text{ for } i = 1.., L.$$

   $\hookrightarrow$ law of demand for inputs...

Properties of cost fct and conditional demand factor fct 5

- $c(.)$ is homogeneous of degree one in $\mathbf{w}$:

$$c(q, \alpha\mathbf{w}) = \alpha\, c(q, \mathbf{w});$$

- $\mathbf{z}(q, \mathbf{w})$ is homogeneous of degree zero in $\mathbf{w}$:

$$\mathbf{z}(q, \alpha\mathbf{w}) = \mathbf{z}(q, \mathbf{w})$$

  - An equally proportional change of all input prices causes an equal change in total cost but not a change in factor demands.
- These two results depend on the Shepard's lemma, being the conditional factor demands the derivative of the cost function.

Properties of cost fct and conditional demand factor fct 6

- $c(.)$ is nondecreasing in $\mathbf{w}$: if $\mathbf{w}' > \mathbf{w}$, then $c(q, \mathbf{w}') \geq c(q, \mathbf{w})$.

  The total cost of producing q can only increase when at least one of the input prices increases.

  This again depends from the Shepard's lemma, since $\frac{\partial c(\mathbf{w}, q)}{\partial w_i} = z_i^*(\mathbf{w}, q) \geq 0$.

Using the cost function

- Using the cost function, we can rewrite the profit maximisation problem as follows

$$\max_{\mathbf{q} \geq 0} \; \mathbf{p} \cdot \mathbf{q} - c(\mathbf{w}, \mathbf{q})$$

  - Since input are optimally chosen, focus is now on the choice of output only !!!

- When the technology is single-output, condition necessary for q^* to be optimal is

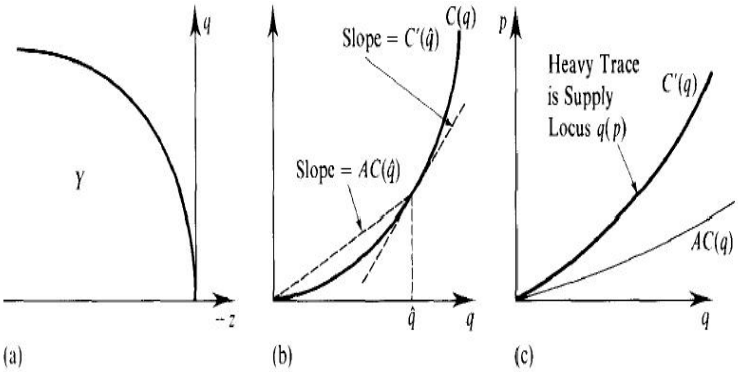
$$p - \frac{\partial c(\mathbf{w}, q^*)}{\partial q} \leq 0 \; \text{ with strict equality if } q^* > 0$$

Competitive firms 1

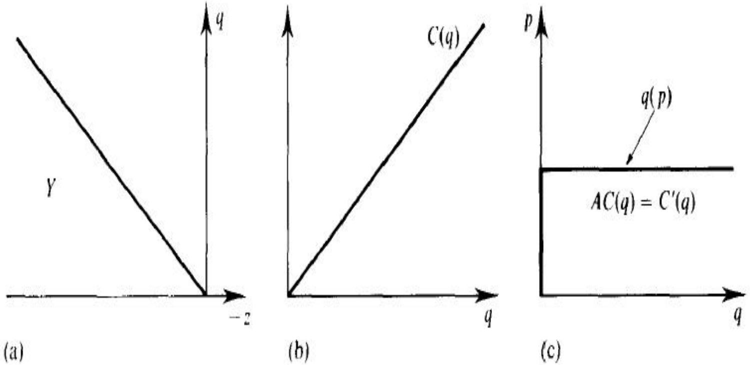
The following figures describe the optimal behaviour of a competitive firm under different technological conditions. Let

- 1 output
- $p > 0$ and $\mathbf{w} \gg 0$
- $C(q) = c(q, \mathbf{w})$;
- $AC(q) = C(q)/q$;
- $C'(q) = dC(q)/dq$

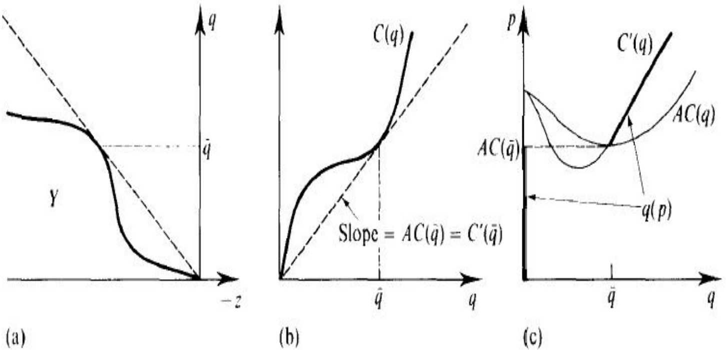
Competitive firms and strictly decreasing returns to scale (convex)



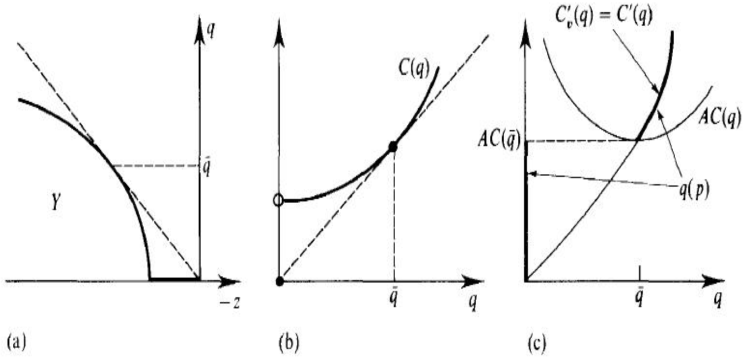
Competitive firms and constant returns to scale (convex)



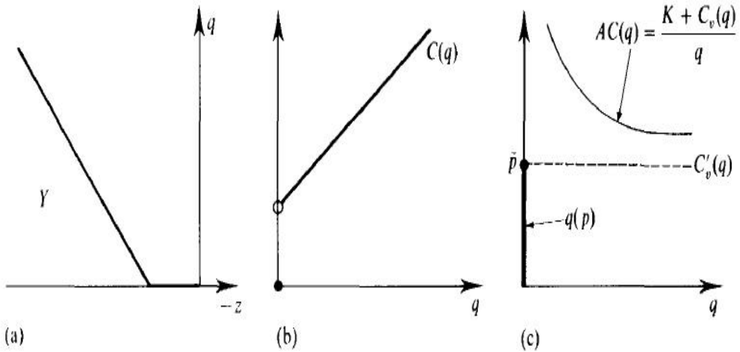
Competitive firms and non convex technology



Competitive firms and strictly convex variable costs with nonsunk setup costs



Competitive firms and constant returns variable costs with nonsunk setup costs



Competitive firms and strictly convex variable costs with sunk setup costs