Choice under uncertainty

Risky alternatives

- Risky alternative are represented by lotteries over a (sometimes) finite number of possible outcomes C
- When outcomes are finite and in number N, a lottery L is a list
$$L = (p_1, \dots, p_N),$$
with $p_n \geq 0$ for all n and $\sum_n p_n = 1$. where p_n is the probability of outcome n to occur.
- outcomes C could be consumption bundles (that is, $C = X$), or other: we'll normally assume that outcomes are monetary payoffs.
- outcomes could also be lotteries themselves: **compound lotteries** (i.e. lotteries over lotteries) could be transformed in reduced lotteries

Preference relation

Our consumer:

- faces a set $\mathcal{L}$ of alternative lotteries
- has a preference relation $\succeq$ over $\mathcal{L}$ which, together with *completeness* and *transitivity*, satisfies
  - *continuity*
    a small change in probability does not change the ranking over lotteries
  - *independence axiom*:
    for all L, L' and $L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, then
    $$L \succeq L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$$
    unlike with "standard" goods, mixing L and L' with L'' does not change our ranking over L and L'
    - either consume L'' (and therefore the ranking over L and L' is not relevant), or
    - consume L and L' (and therefore L'' is irrelevant)

Preference relation and expected utility

If the preference relation $\succeq$ on $\mathcal{L}$ satisfies continuity and independence axioms, can assign an utility number u_n to each outcome so that

- for every simple lottery $L = (p_1, \ldots, p_N) \in \mathcal{L}$, the utility of the lottery, $U(L)$, is the *expected value of the utility of the N outcomes*

$$U(L) = u_1 p_1 + \ldots + u_N p_N$$

  $\hookrightarrow$ <u>von Neumann-Morgestern (VNM) expected utility function</u>

- for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$,

$$L \succeq L' \qquad \Leftrightarrow \qquad \sum_n u_n p_n \geq \sum_n u_n p'_n$$

- if $U(L)$ represents $\succeq$, then this also holds for $V(L) = \gamma + \beta U(L)$ (with $\beta > 0$).

Allais paradox

- So, is everything fine? Well, not quite
- Often behaviour not consistent with EUT

Take the following example:

- Possible outcomes: $c_1 = 2,500,500; c_2 = 500,000; c_3 = 0$
- two sets of lotteries:

$$L_1 = (0; 1; 0) \text{ and } L_1' = (0.1; 0.89; 0.01);$$
$$L_2 = (0; 0.11; 0.89) \text{ and } L_2' = (0.1; 0; 0.9);$$

- now make your choice !!
  - L_1 or L_1' ??;
  - L_2 or L_2' ??;

Allais paradox 2

- Most people show that $L_1 \succ L_1'$ and $L_2' \succ L_2$
- Not consistent with EUT !!
  $L_1 \succ L_1'$ implies
  $u_{05} > (0.10)u_{25} + (0.89)u_{05} + (0.01)u_0$;
  adding $(0.89)u_0 - (0.01)u_{05}$ to both sides, it obtains
  $(0.11)u_{05} + (0.89)u_0 > (0.10)u_{25} + (0.90)u_0$
  that is $L_2 \succ L_2'$
- one solution is *regret theory*: people care not only about what they win but also about what they could have won !
- ... but this is the subject of another course.

Lotteries over monetary outcomes

- Assume outcomes are given by a continuous variable $x \in \Re$.
- Lotteries are then described by the cumulative distribution function $F(.)$ (and the associated density function $f(.)$ where $F(.) = \int_{-\infty}^{x} f(t)dt$).
- A preference relation $\succeq$ with the discussed properites over the set of all possible lotteries ensures that any $F(.)$ (i.e., any lottery) can be evaluated by an utility function of the VNM form
- A VNM expected utility function $U(.)$ is then

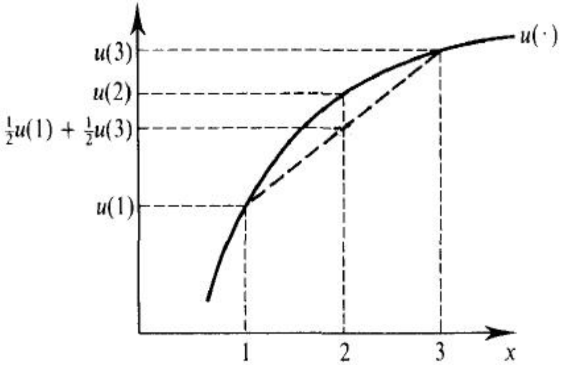
$$U(F) = \int u(x)f(x)dx = \int u(x)dF(x)$$

  where $u(x)$ is the **Bernoulli utility function** which assigns utility to amounts of money.
- Economic attributes of individuals' attitute toward risk are captured by the properties of the Bernoulli function $u(.)$

Risk aversion

A decision maker is *risk averse* iff

$$\int u(x)\, dF(x) \leq u\left(\int x\, dF(x)\right) \text{ for all } F(.) \quad \text{RAD}$$

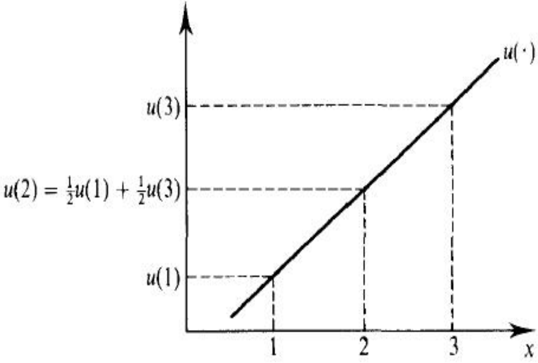


RAD equivalent to concavity of $u(.)$ (strict concavity with strict inequality sign)

Risk neutrality

A decision maker is *risk neutral* iff

$$\int u(x)\,dF(x) = u\left(\int x\,dF(x)\right) \text{ for all } F(.) \quad \text{RND}$$



Two useful concepts

- the **certainty equivalent**
- the **probability premium**

The certainty equivalent

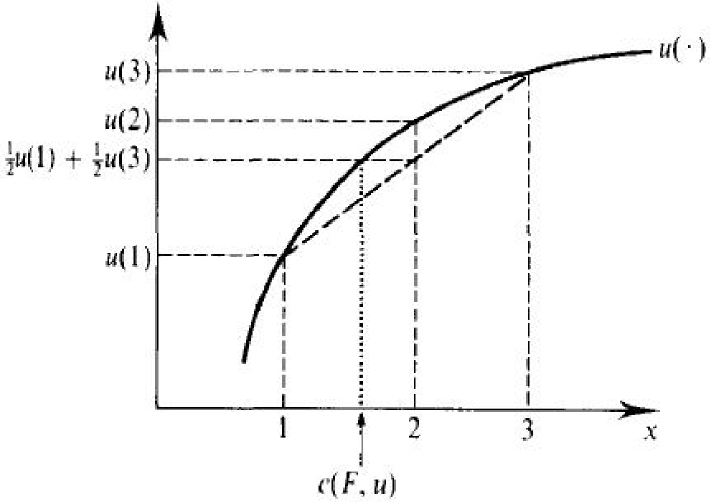
- The *certainty equivalent* of a lottery $F(.)$, denoted with $c(F,u)$ is the amount of money which makes the decision maker indifferent between the money itself and the lottery $F(.)$.
- Formally

$$u(c(F,u)) = \int u(x)\, dF(x)$$

- Risk aversion implies that the certainty equivalent is smaller than the expected value of the lottery:

$$\text{risk aversion} \quad \Leftrightarrow \quad c(F,u) < \int x\, dF(x)$$

Certainty equivalent and risk aversion



The probability premium

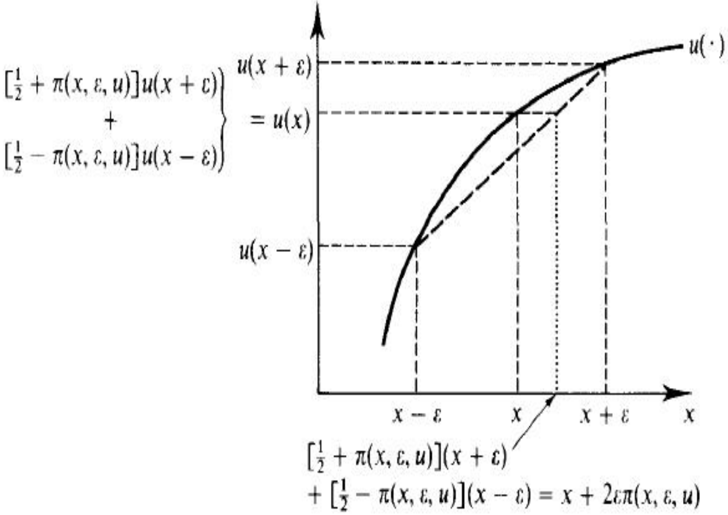
- The *probability premium*, denoted with $\pi(x, \epsilon, u)$ is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome x and a lottery between two equally likely outcomes $x + \epsilon$ and $x - \epsilon$.
- Formally

$$\begin{aligned} u(x) &= \left(\frac{1}{2} + \pi(x, \epsilon, u)\right) u(x + \epsilon) \\ &\quad + \left(\frac{1}{2} - \pi(x, \epsilon, u)\right) u(x - \epsilon) \end{aligned}$$

- Risk aversion implies that the probability premium is positive

$$\text{risk aversion} \quad \Leftrightarrow \quad \pi(x, \epsilon, u) > 0$$

Probability premium and risk aversion



Arrow Pratt coefficient of absolute risk aversion $r_A(.)$

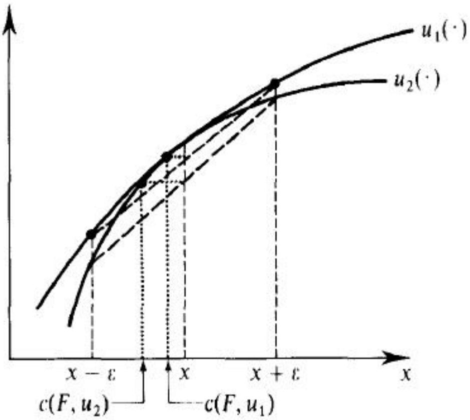
- May be useful to provide a measure of risk aversion.
- $u''(x)$ is a natural candidate; however, its value (but not its sign) depends on the possible linear transformations of $u(.)$
- A measure of absolute risk aversion is given by the

  Arrow Pratt coefficient of absolute risk aversion $r_A(x)$

$$r_A(x; u) = -\frac{u''(x)}{u'(x)}$$

- $r_A(.)$ provides a measure of the curvature of the Bernoulli function
- Notice that integrating twice $r_A(.)$ it is possibile to recover the Bernoulli function

$r_A(.)$ and certainty equivalent



$c(F, u)$ is lower the higher is $r_A(.)$

$r_A(.)$ and probability premium

- Recall the definition of probability premium

$$\begin{aligned} u(x) &= \left(\frac{1}{2} + \pi(x,\epsilon,u)\right) u(x+\epsilon) \\ &\quad + \left(\frac{1}{2} - \pi(x,\epsilon,u)\right) u(x-\epsilon) \end{aligned}$$

- Differentiate twice both sides w.r. to ϵ and evaluate it at $\epsilon = 0$

$$4\frac{\partial \pi}{\partial \epsilon}\Big|_{\epsilon=0}\ u'(x) + u''(x) = 0 \quad \rightarrow \quad r_A(x) = 4\frac{\partial \pi}{\partial \epsilon}\Big|_{\epsilon=0}$$

- $r_A(x)$ then gives the rate at which the probability premium increases with the risk (as measured by ϵ)