The consumer's decision problem

- Let $p \gg 0$ and $w > 0$
- The consumer's decision problem may be described as

$$\max_{x \geq 0} \; u(x) \quad \text{s.t. } p \cdot x \leq w \qquad \text{UMP}$$

- In words:

choose the preferred consumption bundle
within the set of admissible bundles

UMP

From UMP, two interesting objects:

- optimal consumption bundles: the solution to UMP
- consumer's maximal utility value: the value function of the UMP

UMP

From UMP, two interesting objects:

- ↪ optimal consumption bundles: the solution to UMP
- consumer's maximal utility value: the value function of the UMP

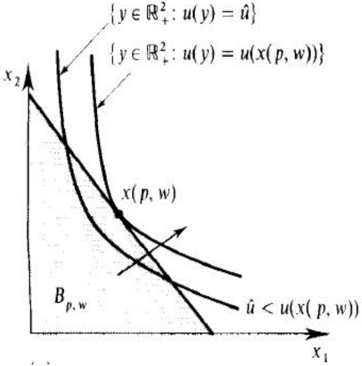
The solution to UMP

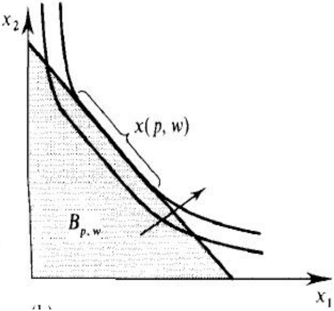
The solution to the UMP is the Walrasian (or ordinary or market or Marshallian) demand correspondence (or function): $x(p, w)$

a rule that assigns optimal consumption vector(s)
to each price-wealth combination

If $x(p, w)$ is single valued, then *Walrasian demand function*, otherwise *Walrasian demand correspondence*

The solution to UMP





Properties of Walrasian demand correspondence

1) homogeneity of degree zero in p and w:

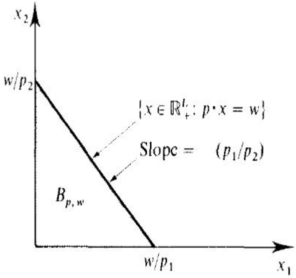
$$x(p, w) = x(\alpha p, \alpha w) \text{ for any } \alpha > 0$$

All is due to the budget set being unaffected by a proportional changes in prices and income/wealth.
Let

- $B_{p,w} = \{x \in \Re^L_+ : p \cdot x \leq w\}$
- $B_{\alpha p,\alpha w} = \{x \in \Re^L_+ : \alpha p \cdot x \leq \alpha w\}$.

Then

$$B_{p,w} = B_{\alpha p,\alpha w}$$



Properties of Walrasian demand correspondence (ctd)

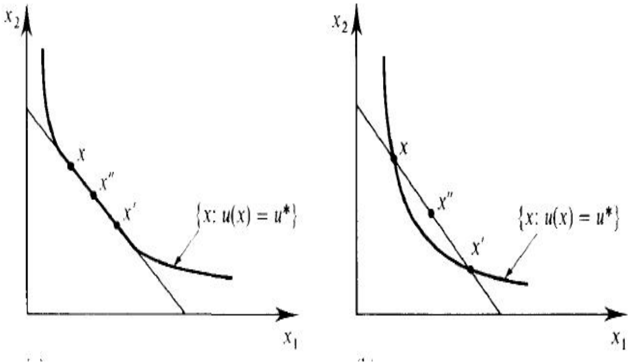
2) Walras's law

$$w \equiv p \cdot x(p, w)$$

It follows from *local non satiation*: if the consumer selects a consumption bundle x where $p \cdot w < w$, then there must be another consumption bundle y, ϵ-close to x and affordable (i.e., such that $p \cdot w \leq w$) where the consumer can improve its utility

Properties of Walrasian demand correspondence (ctd)

3) $x(p, w)$ is convex
   - if $\succeq$ is convex (i.e. $u(x)$ is quasi-concave), then $x(p, w)$ is a convex set
   - if $\succeq$ is strictly convex (i.e. $u(x)$ is strictly quasi-concave), then $x(p, w)$ is a single element



Necessary conditions for solution to UMP

If $x^* \in x(p, w)$ is a solution to UMP, then there exists a Lagrangean multiplier λ such that, for all ℓ,

$$\frac{\partial u(x^*)}{\partial x^*_\ell} \leq \lambda^* p_\ell$$
$$\left(\frac{\partial u(x^*)}{\partial x^*_\ell} - \lambda^* p_\ell\right) x^*_\ell = 0 \qquad \text{FOC-UMP}$$

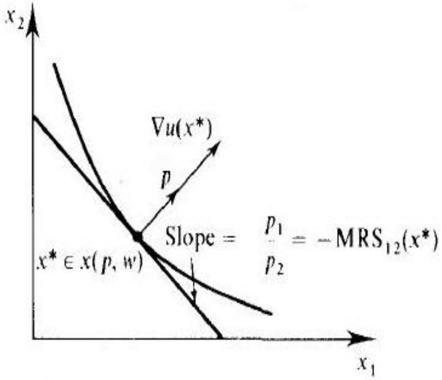
In matrix notation

$$\nabla u(x^*) \leq \lambda^* p \qquad x^* \cdot (\nabla u(x^*) - \lambda^* p) = 0$$

FOC-UMP are sufficient for global maximum if

- $u(.)$ is quasiconcave
- $\nabla u(x) \neq 0$ for all $x \in \Re^L_+$ (otherwise, 'bliss' point)

FOC-UMP



Marginal rate of substitution

FOC-UMP imply that, for any ℓ, k, at an interior solution

$$\underbrace{\frac{\frac{\partial u(x^*)}{\partial x_\ell^*}}{\frac{\partial u(x^*)}{\partial x_k^*}}}_{MRS_{\ell,k}} = \underbrace{\frac{p_\ell}{p_k}}_{\textit{price ratio}}$$

- $MRS_{\ell,k}$ tells how much you want, at the margin, to trade good ℓ for good k and keep utility constant.

If above equality not satisfied,

$$\frac{\frac{\partial u(x^*)}{\partial x_\ell^*}}{p_\ell} > \frac{\frac{\partial u(x^*)}{\partial x_k^*}}{p_k}$$

- The MU per dollar spend on good ℓ is larger that the MU per dollar spent on good k: consumer would like to increase his/her consumption of good ℓ even more
- Trading commodities ℓ and k at current prices increases utility, contradicting the maximality of x^*

Marginal rate of substitution

FOC-UMP imply that, at a corner solution

- $\frac{\partial u(x^*)}{\partial x_k^*} \leq \lambda p_k$ for those goods such that $x_k^* = 0$
- $\frac{\partial u(x^*)}{\partial x_\ell^*} = \lambda p_\ell$ for those goods such that $x_\ell^* > 0$
- so that

$$\frac{\frac{\partial u(x^*)}{\partial x_\ell^*}}{p_\ell} = \lambda > \frac{\frac{\partial u(x^*)}{\partial x_k^*}}{p_k}$$

The MU per dollar spend on good ℓ is larger that the MU per dollar spent on good k: consumer would like to increase his/her consumption of good ℓ even more, but he/she cannot!!

Lagrange multiplier in UMP

Lagrange multiplier λ gives the shadow value of the constraint

λ gives the change in (optimal) utility
due to a change in wealth

At an interior solution,

$$\begin{aligned}
\frac{\partial u(x^*)}{\partial w} &= \sum_\ell \frac{\partial u(x^*)}{\partial x_\ell^*}\frac{\partial x_\ell^*}{\partial w} && \text{using } u(x^*) = u(x(p,w)) \\
&= \sum_\ell \lambda^* p_\ell \frac{\partial x_\ell^*}{\partial w} && \text{using FOC-UMP} \\
&= \lambda^* \sum_\ell p_\ell \frac{\partial x_\ell^*}{\partial w} \\
&= \lambda^* && \text{by differentiating Walras's law}
\end{aligned}$$

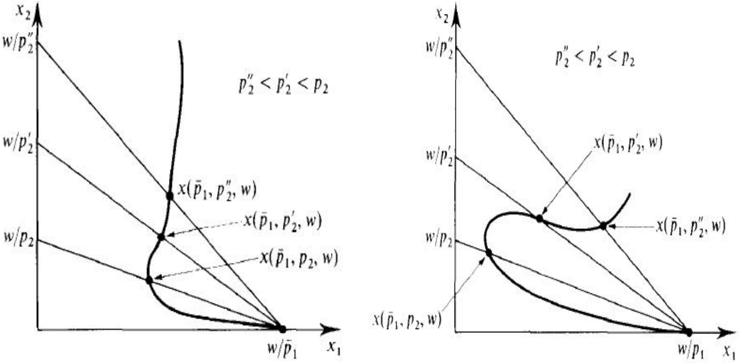
Comparative statics

- Useful to look at how demand $x(p, w)$ changes when w or p change
- We'll look at two objects:
  - *price/wealth effects*: the effect on $x_\ell(p, w)$ of a change in p_ℓ or w.
  - *Engle/offer function*: optimal bundle $x(p, w)$ as a function of w or p_ℓ.
- Helpful to have Walrasian demand be continuous and differentiable. Possible to establish that, when preferences are continuous, strictly convex, locally nonsatiated on the consumption set $\Re^L_+$, the function $x(p, w)$ is continuous for all $(p, w) \gg 0$.

Comparative statics: price effects

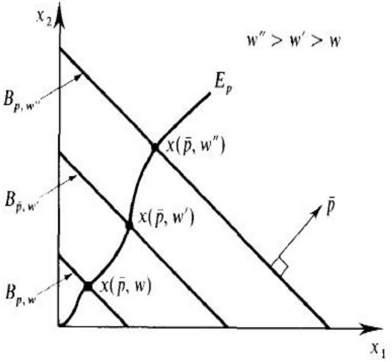
- The price offer curve $x(\overline{p}_1, \ldots, p_\ell, \ldots, \overline{p}_L, \overline{w})$ gives the optimal bundle as a function of p_ℓ, for given prices $p_{-\ell}$ and wealth w.
- The price effect of p_k on the demand for good ℓ is given by the derivative $\frac{\partial x_\ell(p,w)}{\partial p_k}$.
- Much more on this later...
- For the moment, satisfy yourself with the following
  - own-price effect: $\frac{\partial x_\ell(p,w)}{\partial p_\ell}$
    - when $\frac{\partial x_\ell(p,w)}{\partial p_\ell} > 0$, ℓ is a *Giffen good* (at the current price-wealth combination)
  - cross-price effect: $\frac{\partial x_\ell(p,w)}{\partial p_k}$
    - when $\frac{\partial x_\ell(p,w)}{\partial p_k} > 0$, ℓ and k are substitutes
    - when $\frac{\partial x_\ell(p,w)}{\partial p_k} < 0$, ℓ and k are complements

Comparative statics: price effects



Comparative statics: wealth effects 1

- For given prices $\overline{p}$, the Engel function $x(\overline{p}, w)$ gives the optimal bundle as a function of wealth, for given prices.
- It can be represented by the *wealth expansion path* $E_p = \{x(p, w) : w > 0\}$



Comparative statics: wealth effects 2

The <u>wealth effect</u> for commodity ℓ is given by $\frac{\partial x_\ell(p,w)}{\partial w}$.

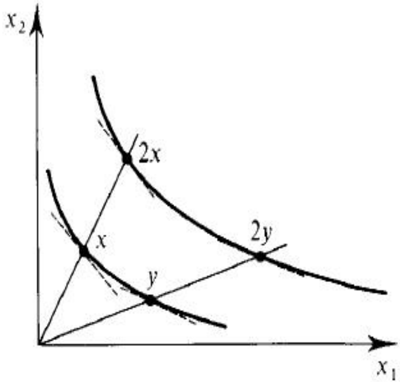
- commodity ℓ is *normal* at (p, w) if its demand increases with wealth
$$\frac{\partial x_\ell(p,w)}{\partial w} \geq 0 \quad \Leftrightarrow \quad \text{NORMAL GOOD}$$
- commodity ℓ is *inferior* at (p, w) if its demand decreases with wealth
$$\frac{\partial x_\ell(p,w)}{\partial w} < 0 \quad \Leftrightarrow \quad \text{INFERIOR GOOD}$$

Wealth effect for special preferences

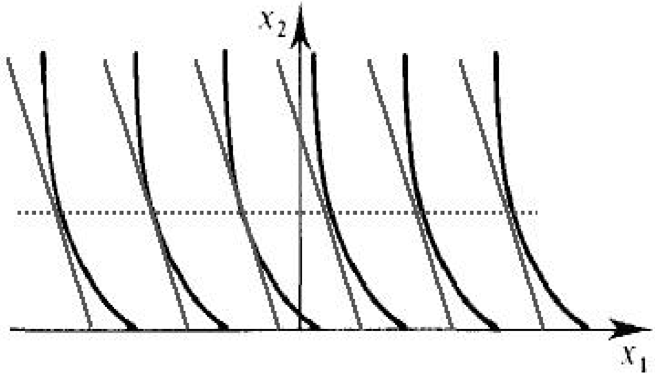
The wealth effect typically depends on the level of w.
However, for some types of preferences, this is not so:

- when preferences are *homothetic*, the wealth expansion path is a straight line through the origin
  $\hookrightarrow$the wealth effect is constant
- when preferences are *quasi-linear*, the wealth expansion path is a straight horizontal line
  $\hookrightarrow$the wealth effect is zero

Engel function for homothetic preferences



Engel function for quasi-linear preferences



UMP

From UMP, two interesting objects:

- optimal consumption bundles: the solution to UMP

$\hookrightarrow$ consumer's maximal utility value: the value function of the UMP

The indirect utility function

The value function of the UMP is called the *indirect utility function* $v(p, w)$

$$v(p, w) = u(x^*) = u(x(p, w))$$

Properties of the indirect utility function:

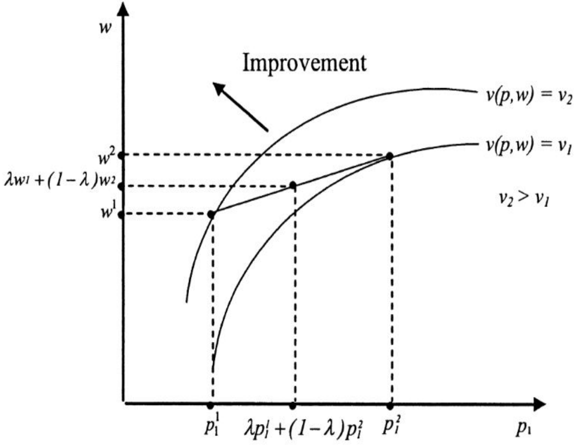
1. homogeneous of degree zero in p and w

$$v(p, w) = v(\alpha p, \alpha w) \text{ for any } \alpha > 0$$

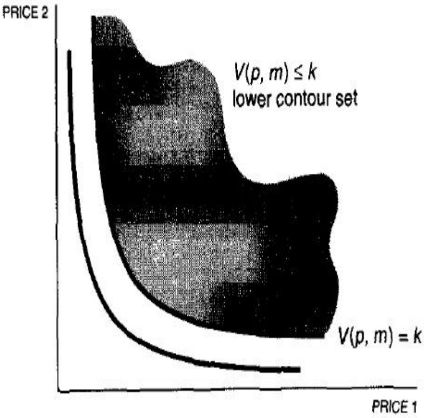
2. strictly increasing in w and non increasing in p_ℓ
3. continuous in p and w
4. quasi-convex: the set $\{v(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$

the lower contour set is convex

Quasi-convexity of $v(p, w)$

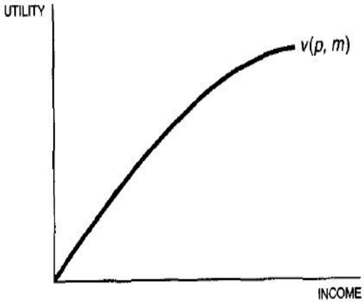


Price indifference curves



Inverting the indirect utility function

Since strictly increasing and continuous in w, can invert $v(p, w)$ to give the minimum level of income necessary to reach a given level of utility.



Formally, this may be stated as the ...

The expenditure minimisation problem

The consumer's decision problem may also be described as

$$\min_{x \geq 0} \; p \cdot x \quad \text{s.t.:} u \geq \overline{u} \qquad \mathrm{EMP}$$

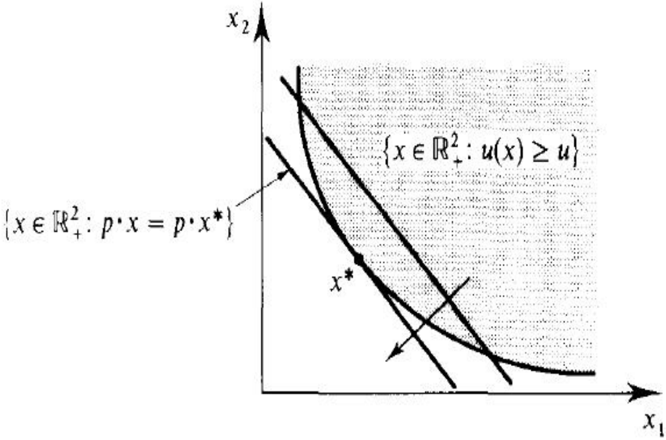
In words,

- choose the least-cost consumption bundle
which ensures utility $\overline{u}$

EMP is the *dual* problem of UMP

- it reverses the role of the objective function and of the constraint

The expenditure minimisation problem

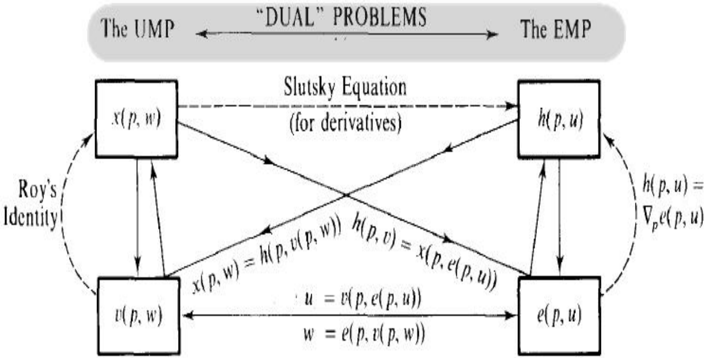


Relationship between EMP e UMP

Strict relationship between EMP e UMP
For a given p

- if x^* solves UMP for $\tilde{w}$, then x^* solves EMP for $u = u(x^*)$: at x^*, total expenditure is $p \cdot x^* = \tilde{w}$
- if x^* solves EMP for $\tilde{u}$, then x^* solves UMP when $p \cdot x^* = w$; at x^*, utility is $\tilde{u}$

Moving around objects



EMP

From EMP, two interesting objects:

- optimal consumption bundles: the solution to EMP
- consumer's minimal expenditure: the value function of the EMP

EMP

From EMP, two interesting objects:

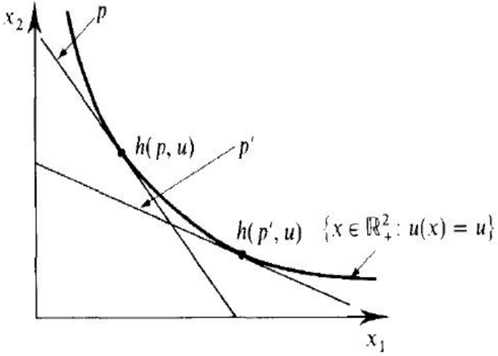
- $\hookrightarrow$ optimal consumption bundles: the solution to EMP
- consumer's minimal expenditure: the value function of the EMP

Necessary conditions for solution to EMP

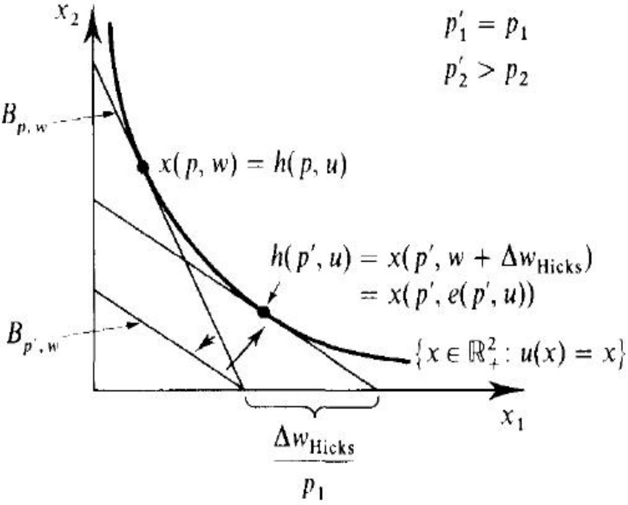
- The set of optimal commodity vector in EMP is $h(p, u)$ and is known as Hicksian (or compensated) demand correspondence (or function)
- If $h^* \in h(p, u)$ is a solution to EMP, then there exists a Lagrangean multiplier λ such that

$$p \geq \lambda \nabla u(h^*) \qquad h^* \cdot (p - \lambda \nabla u(h^*)) = 0$$

Hicksian demand



Why "compensated"??



Properties of Hicksian demand correspondence

Properties of Hicksian demand correspondence are

1. homogeneity of degree zero in p:
$$h(\alpha p, u) = h(p, u) \text{ for any } \alpha > 0$$
2. no excess utility: for any $x \in h(p, u)$, then $u(x) = u$
3. $h(p, u)$ is convex
   - if $\succeq$ is convex (i.e. $u(x)$ is quasi-concave), then $h(p, u)$ is a convex set
   - if $\succeq$ is strictly convex (i.e. $u(x)$ is strictly quasi-concave), then $h(p, u)$ is a single element

Compensated law of demand

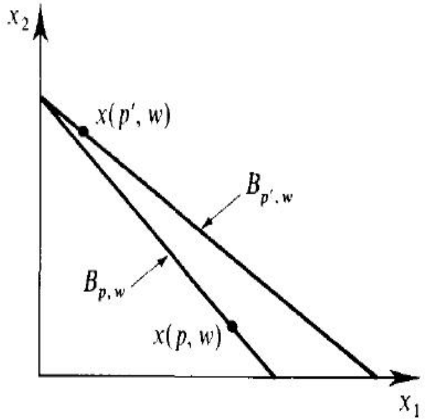
For all p' and $p'' \gg 0$,

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0$$

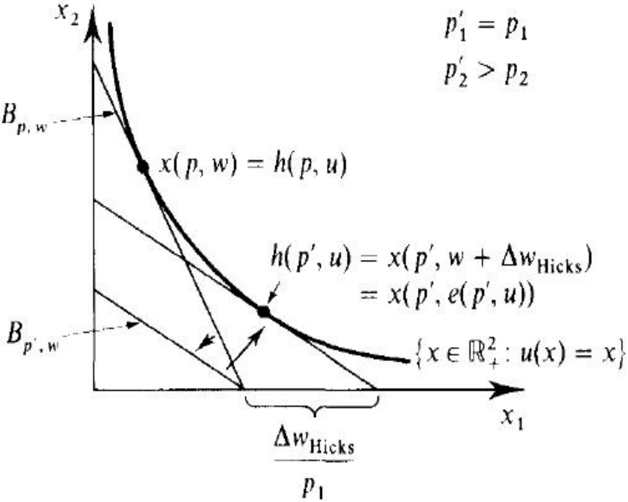
$\Rightarrow$ own-price effects are non positive

Always true for Hicksian demand but not necessarily the case for Walrasian demand

Law of demand for Walrasian demand



Law of demand for Hicksian demand



Relationship between Walrasian and Hicksian demand

Clear relationship between Walrasian and Hicksian demand correspondence

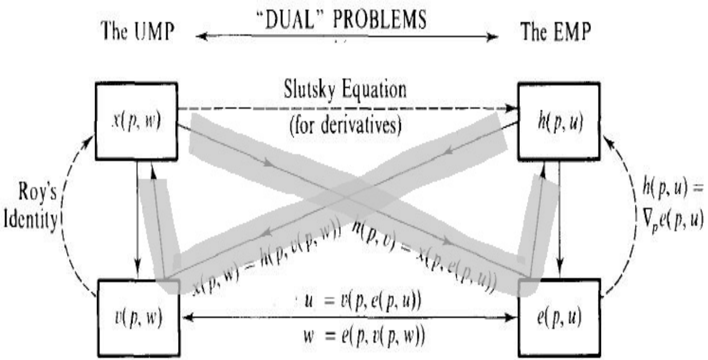
$$x(p, w) = x(p, e(p, u)) = h(p, u)$$

and

$$h(p, u) = h(p, v(p, w)) = x(p, w)$$

These relationships have an operational content
$\hookrightarrow$ how to obtain one type of demand from the other

Moving around objects



EMP

From EMP, two interesting objects:

- optimal consumption bundles: the solution to EMP

$\hookrightarrow$ consumer's minimal expenditure: the value function of the EMP

The expenditure function

The value function of the EMP is called the *expenditure function* $e(p, u)$.
If denote with x^* any solution to EMP, then

$$e(p, u) = p \cdot x^*$$

Properties of the expenditure function (analogous to those of the indirect utility function)

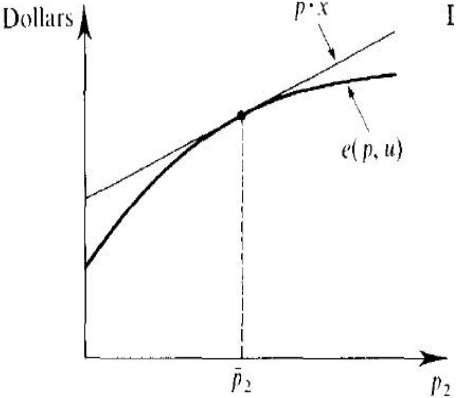
1. homogeneous of degree 1 in p: $e(\alpha p, u) = \alpha e(p, u)$ for any $\alpha > 0$
2. strictly increasing in u and non decreasing in p_ℓ
3. continuous in p and u
4. concave in p

Concavity of the expenditure function

- Most important property of the expenditure function is the concavity in prices
- Intuition: Let $\overline{x}$ solve EMP when prices are $\overline{p}$ and welath is $\overline{w}$ If p_1 changes and $\overline{x}$ doesn't
  - total expenditure varies linearly with p_1
  - this "passive" total expenditure has to be higher than (or equal to) total expenditure with optimal behaviour (as from $e(p, u)$)

$$p \cdot \overline{x}(\overline{p}, \overline{w}) \geq \overline{p} \cdot \overline{x}(\overline{p}, \overline{w}) = e(\overline{p}, u)$$

Concavity of the expenditure function



Relationship between expenditure and indirect utility functions

The relation between the solutions of the UMP and the EMP implies that

$$e(p, v(p, w)) = w \text{ and } v(p, e(p, u)) = u$$

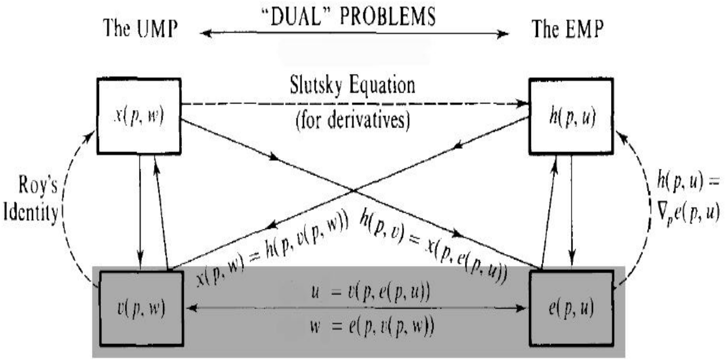
Further consequence is that, for a given price vector

$$e(p, u) \text{ and } v(p, w) \text{ are inverses to one another}$$

In other words, can solve

- $w = e(p, u)$ for u to have the indirect utility fct $v(p, w)$
- $u = v(p, w)$ for w to have the expenditure fct $e(p, u)$

Moving around objects



Relationships between objects

On further existing relationships between the different objects studied so far

- Hicksian demand and expenditure function
- Hicksian demand and Walrasian demand
- Walrasian demand and indirect utility function

Relationships between objects

On the existing relationships between the different objects studied so far

- $\hookrightarrow$ Hicksian demand and expenditure function
- Hicksian demand and Walrasian demand
- Walrasian demand and indirect utility function

Hicksian demand and expenditure function

Easy to go from *Hicksian demand* to the *expenditure function*:

$$p \cdot h(p, u) = e(p, u)$$

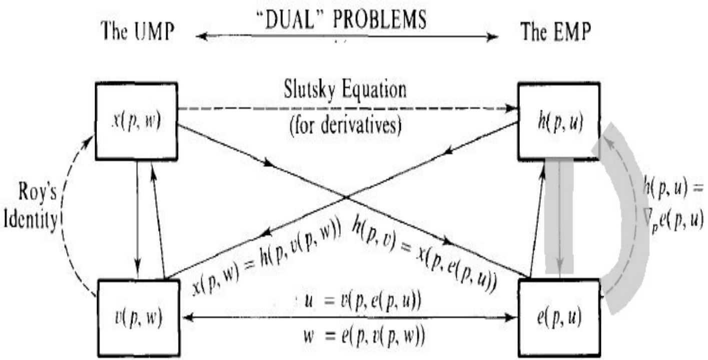
But equally easy to do the opposite:

$h(p, u) = \nabla_p e(p, u)$ SHEPARD'S LEMMA

that is

$h_\ell(p, u) = \frac{\partial e(p,u)}{\partial p_\ell}$ for all ℓ

Moving around objects (more)



Shepard's lemma 1

Notice that

$$
\begin{aligned}
\nabla_p e(.) &= \nabla_p [p \cdot h(.)] && \text{by the defn of } e(.) \\
&= h(p,u) + [p \; D_p h(.)]^T && \text{by the chain rule} \\
&= h(.) && \text{using } p = \lambda \nabla u(h(.)) \\
&+ \lambda [\nabla u(h(.)) \; D_p h(.)]^T && \text{from FOC-UMP} \\
&= h(p,u) && \text{differenting w. r. to } p \\
& && \text{constraint in EMP} \\
& && u(h(p,u)) = \overline{u}
\end{aligned}
$$

Shepard's lemma 2

In words,

a change in prices has two effects on optimal total expenditure

- $h(p, u)$ direct effect holding demand fixed
- $[p\ D_p h(p, u)]$ indirect effect due to the induced change in demand holding prices fixed

The second indirect effect cancels out since bundles always minimise costs

Price derivatives of Hicksian demand

- The relationship between Hicksian demand and expenditure function has consequences on the matrix of price derivatives of Hicksian demand.
- Let $D_p h(p, u)$ denote the $L \times L$ matrix of first price derivative of Hicksian demands. Then
  1. $D_p h(p, u) = D_p^2 e(p, u)$
  2. $D_p h(p, u)$ is symmetric
  3. $D_p h(p, u)$ is negative semi-definite

     A matrix is Negative Semi-definite if the determinants of all of its principal submatrices are alternate in sign, starting with a negative (with the allowance here of 0 determinants replacing one or more of the positive or negative values)
- 1. and 2. are natural consequences of the Shepard's lemma;
- 3. follows from the concavity of $e(p, u)$ and implies the *compensated law of demand*

Relationships between objects

On the existing relationships between the different objects studied so far

- Hicksian demand and expenditure function

$\hookrightarrow$ Hicksian and Walrasian demand

- Walrasian demand and indirect utility function

Hicksian and Walrasian demand

$h(.)$ not observable, so?

Can obtain its derivatives from the observable $x(p, w)$ using the

$\Rightarrow$*Slutsky equation*$\Leftarrow$

$$\frac{\partial h_\ell(p, u)}{\partial p_k} = \frac{\partial x_\ell(p, w)}{\partial p_k} + \frac{\partial x_\ell(p, w)}{\partial w} x_k(p, w)$$

Easy to obtain, simply differentiating w.r.to p_k the equality

$$h_\ell(p, u) = x_\ell(p, e(p, u))$$

and using Shepard's lemma.

Consequences of Slutksy equation 1

Some important consequences of Slutksy equation

- Hicksian demand "steeper" than Walrasian demand when commodities are normal
- matrix $D_p\, h(p,u)$ observable

$$D_p\, h(p,u) = S(p,w) = \begin{pmatrix} S_{11} & \dots & S_{L1} \\ \vdots & \ddots & \vdots \\ S_{1L} & \dots & S_{LL} \end{pmatrix}$$

Consequences of Slutksy equation 2

Some important consequences of Slutksy equation

- law of demand not valid for Walrasian demand. Rearranging Slutsky eqn as

$$\underbrace{\frac{\partial x_\ell(p,w)}{\partial p_\ell}}_{(+/-)} = \underbrace{\frac{\partial h_\ell(p,u)}{\partial p_\ell}}_{(-)} - \underbrace{\frac{\partial x_\ell(p,w)}{\partial w} x_\ell(p,w)}_{\substack{(+)\text{for normal goods} \\ (-)\text{for inferior goods}}}$$

If good ℓ is "highly" inferior, 2nd term outplays first term and $\frac{\partial x_\ell(p,w)}{\partial p_\ell}$ becomes positive

a Giffen good must be inferior, but not viceversa

Relationships between objects

On the existing relationships between the different objects studied so far

- Hicksian demand and expenditure function
- Hicksian and Walrasian demand

$\hookrightarrow$ Walrasian demand and indirect utility function

Roys' identity

Easy to go from *Walrasian demand* to the *indirect utility function*

$$v(p, w) = u(x^*) = u(x(p, w), w)$$

But (almost) equally easy to do the opposite. For all ℓ,

$$x_\ell(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_\ell}}{\frac{\partial v(p,w)}{\partial w}} \qquad \text{ROY'S IDENTITY}$$

Easy to obtain, simply differentiating w.r.to p_ℓ the equality $v(p, e(p, u)) = u$

Moving around objects (more)