Topic II Optimal inter-temporal consumption and growth.

Today we will discuss the Ramsey-Cass-Koopmans model (again). This is the first topic of Romer chapter II. The idea is to combine the production sector from Solow (with technological progress and no adjustment costs) with consumers who maximize lifetime utility.

Some Review of Lecture 1

Recall the Assumptions that output is a function of capital K, employment L, and human capital or labour augmenting (Harrod Neutral) technology A. L is effective labour equal to A times L

1) $Y = F(K,AL)$

F is an aggregate production function with constant returns to scale and decreasing marginal product of capital and labour.
Today, following Romer , assume that capital doesn't depreciate so $\delta = 0$. Assume perfect competition so labour and capital are paid their marginal products. Thus $r_t = F_K(K_t, A_tL_t)$. Perfect competition and constant returns to scale imply that

2) $W_tL_t = Y_t - K_tr_t$

I assume that $L^{dot}{}_t = nL_t$ where n is the rate of growth of log employment and that $A^{dot}{}_t = gA_t$ so g is the rate of labour augmenting technological progress.

Now define $k_t = K_t/(A_tL_t)$ $y_t = Y_t/(A_tL_t)$ $c_t = \text{Totalconsumption}_t/(A_tL_t)$ and $w_t=W_t/A_t$ Given the assumption of constant returns to scale $F(K_t, A_tL_t) = A_tL_tF(k_t,1)$. Define $f(k_t) = F(K_t,L_t)/(A_tL_t)= F(k_t,1)$. This means that $y_t = f(k_t)$. Note that

3) $r_t = F_K(K_t,A_tL_t) = A_tL_tF_K(k_t,1) = A_tL_t df(K_t/L_t)/dK_t = f'(k_t)$

and

4) $w_t = f(k_t) - k_tf'(k_t)$.

And

5) $k_t^{dot} = f(k_t) - c_t - (n + g)k_t$

This describes the options open to society.

End of review

To consider consumption savings choices, we really have to divide workers into households since consumption is a really done by a household. (Households are groups of people who live together and share living expenses so they are like families except that we don't care if couples are technically married because we are economists not priests or tax collectors). This is also helpful, because we can assume that households last forever and we can even assume that the number of households is a constant H (as usual we can assume things that aren't true. As noted above, we are economists).

The number of worker/consumers per household L/H grows at rate n. C_p is equal to household consumption per person so it is total consumption divided by L (this is C according to Romer).

$$6)\ \max V = \int_0^\infty e^{-\rho t} u(C_{pt})(L_t/H)dt =$$

$$(L_o/H) \int_0^\infty e^{(n-\rho)(t)} u(C_{pt})dt$$

For g=0, this implies a steady state capital labour ratio described by

$$7)\ f'(k) = \rho$$

This implies that if consumers are more impatient they obtain lower steady state consumption level in exchange for more consumption now.

$$\text{Define } R_t = \int_0^t r_s\, ds$$

Each household has a budget constraint that the present value of consumption discounted to time 0 is equal to initial wealth plus the present value of labour income discounted to time 0

8) $$\int_0^{\infty} e^{-R_t} C_{pt}(L_t/H)dt \le K_o/H + \int_0^{\infty} e^{-R_t} W_t(L_t/H)dt$$

Optimal choice implies that this equation holds with equality (no slack - no way to consume more now without consuming less later)

9) $$\int_0^{\infty} e^{-R_t} C_{pt}(L_t/H)dt = K_o/H + \int_0^{\infty} e^{-R_t} W_t(L_t/H)dt$$

Maximization of V subject to 9 is equivalent to unconstrained maximization of

10

$$\int_0^{\infty} e^{-\rho t} u(C_{pt})(L_t/H) \, dt -$$

$$\lambda\left[\int_0^{\infty} e^{-R_t} C_{pt} (L_t/H) \, dt - K_o/H - \int_0^{\infty} e^{-R_t} W_t(L_t/H)dt \right]$$

For some lagrange multiplier λ

For each t there is a first order condition for C_{pt}

11) $e^{-\rho t} u'(C_{pt})(L_t/H) - e^{-Rt}(L_t/H)$

So

12) $e^{-\rho t} u'(C_{pt}) = e^{-Rt}$

So

13) $\ln(u'(C_{pt})) = \rho t - R_t$

So

14) $d(\ln(u'(C_{pt})))/dt = \rho - r_t$

For positive g, it is necessary to make specific assumptions about u in order to figure out what would happen.

Assume CES utility so

15) $u(C_p) = C_p^{1-\theta}/(1-\theta)$

and

16) $u'(C_p) = C_p^{-\theta}$

So

17) $\ln(u'(C_p) = -\theta \ln(C_p)$

14 becomes

18) $-\theta(d(C_{pt})/dt)/ C_{pt} = \rho - r_t$

So

19) $((dC_{pt}/dt)/ C_{pt} = (r_t - \rho)/\theta$

We're not done yet. To draw a phase diagram, we have to get to dc_t/dt where $c = C_p/A$. It helps to take logs since

$((dC_{pt}/dt)/ C_{pt} = d(\ln C_{pt} /dt)$ and

$\text{Ln}(c_t) = \ln(C_{pt}) - \ln(A_t) = \ln(Cp) - \text{Ln}(A_o) - gt$

So

20) $d\text{Ln}(c_t)/dt = ((r-\rho)/\theta) - g = (r-\rho-\theta g)/\theta$

And with 5 and 20 we can draw a phase diagram.

Now imagine that time comes in multiples of d so t is an element of d,2d,3d,4d …

Discrete time is a hassle.

It's all about the beginning of period t and the end of period t. We have production in the first part of period t, then people are paid wages and interest then they decide how much to consume and save. Their wealth at the end of the period is their wealth at the end of last period plus income minus consumption. To avoid driving myself crazy, we are all one big happy family so H = 1 but I will assume that this household takes W and r as given.

$C_t d$ is total consumption in period t, $C_{pt}d$ is $C_t d/L_t$ and c_t is, as always, $C_t/(A_t L_t)$

Everyone works. They get $W_t d$ in time t so W is the wage per unit of time worked. If I own wealth B at the end of time t-d then I have $B(1+r_t d)$ at time t. For d=1 this is the notation I used last semester.

14) $L_{t+d} = L_t(1+nd)$

Total capital follows the rule

15) $K_{t+d} = W_t L_t d + K_t(1+r_t d) - C_t d =$

$W_t L_t d + K_t(1+r_t d) - C_{pt} L_t d$

This means K_t is capital at the beginning of time t

Households (whose number I am setting at 1 to avoid messing up) maximize

$$15)\ V_1 = \sum_{i=1}^{\infty} e^{-\rho di} u(C_{pt}) L_t$$

OK now let's figure out optimal consumption telling a story, that is not by appealing to Lagrange or Hamilton or any of them.

For the optimal consumption plan C if a household increases C_t by x and reduce C_{t+d} by $x(1+r_t d)$ so that the new K_{t+2d}
Is the same as in the old optimal plan, then the derivative of V_1 with respect to x has to be zero at x = 0. Note that

16) $dC_{pt}/dC_t = 1/L_t$

So for all i

17) $e^{-\rho di} u'(C_{pt}) - (1+r_t d) e^{-\rho d(i+1)} u'(C_{pt+d}) = 0$

So

18) $u'(C_{pt}) - (1+r_t d) e^{-\rho d} u'(C_{pt+d}) = 0$

For $d = 1$, this is very similar to the equation from last semester except that I have rewritten the discount factor.

OK now an approximation for small d, $\ln(1+r_t d) \approx r_t d$
and $(e^{-\rho d} -1)/d \approx \rho$ and $(e^{-rd}-1)/d \approx r$
so $((1+r_t d)e^{-\rho d} - 1)/d \approx r-\rho$

Now **assume** that for small enough d C_{pt+d} is small (this is an assumption). Then 18 becomes

19) $u'(C_{pt})-(1+r_t d)e^{-\rho d}\,[u'(C_{pt}) + u''(C_{pt})(\,C_{pt+d} - C_{pt})] \approx 0 \approx$

$u'(C_{pt})(1 - (1+ r_t d - \rho d - r_t\rho d^2) +$
$(1+r_t d)e^{-\rho d}\,[\,u''(C_{pt})(\,C_{pt+d} - C_{pt})] \approx$
$[(\rho-r_t)u'(C_{pt})- u''(C_{pt})(\,C_{pt+d} - C_{pt})/d]d$

As d goes to zero this implies
$(\rho-r_t)u'(C_{pt})- u''(C_{pt})[d(C_{pt})/dt] = 0.$

Notice first that that's how it was done and second that there is an assumption -- basically that C_{pt} is a continuous function of time. The assumption is really needed.