# Choice under Uncertainty

## 1. Primitives

Finite set $A = \{a_1, \dots, a_n\}$ of outcomes. A simple gamble (or lottery) is an element of $\Delta(A)$, the set of probability distributions over $A$. We denote the set of simple gambles by $\mathcal{G}_s$ and a typical element of this set by $g = (p_1 \circ a_1, \dots, p_n \circ a_n)$, where $(p_1, \dots, p_n)$ is an element of the unit simplex in $\mathbb{R}^n$; i.e., $p_i \geq 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n p_i = 1$. When $p_i = 0$, we drop the term $p_i \circ a_i$ from the lottery $g$. The probabilities in this context are *objective*.

Not all gambles are simple. For example, if $A = \{-1, 0, 1\}$ and $\widetilde{g} = \left(\frac{1}{2} \circ -1, \frac{1}{2} \circ 1\right)$, then $g = \left(\frac{1}{2} \circ 0, \frac{1}{2} \circ \widetilde{g}\right)$ is also a gamble. The gamble $g$ is an example of a compound gamble. We denote the set of all compound gambles by $\mathcal{G}$ and also use $g$ to denote a typical element of this set. Every compound gamble $g$ uniquely defines a probability distribution $(p_1, \dots, p_n)$ over $A$. Let $g_s = (p_1 \circ a_1, \dots, p_n \circ a_n)$. We refer to $g_s$ as the simple gamble associated to $g$. In the example above, $g_s = \left(\frac{1}{4} \circ -1, \frac{1}{2} \circ 0, \frac{1}{4} \circ 1\right)$.

The choice set is the set $\mathcal{G}$ of compound gambles.

## 2. Preferences

As in the consumer theory, we represent preferences over gambles by a binary relation $\succsim$ on the set $\mathcal{G}$. We make the following assumptions about $\succsim$.

**1.** *Completeness:* $\succsim$ is complete.

**2.** *Transitivity:* $\succsim$ is transitive.

The first two properties imply that: *(i)* the decision maker (agent) is able to compare any two given gambles; and *(ii)* the decision maker's ranking of lotteries is internally consistent. As in the consumer theory, we say that $\succsim$ is a preference relation if it is both complete and transitive.

The preference relation $\succsim$ on $\mathcal{G}$ induces an order on $A$, that we also denote by $\succsim$, as follows: $a_i \succsim a_j$ if, and only if, $1 \circ a_i \succsim 1 \circ a_j$. In what follows we denote the lottery $1 \circ a_i$ by $a_i$, thus erasing any distinction between preferences on $\mathcal{G}$ and preferences on $A$. Since $\succsim$

is complete, we can rank all elements of $A$. Assume, without loss, that $a_1 \succsim \ldots \succsim a_n$.

**3.** *Continuity:* for any $g \in \mathcal{G}$, there exists $\alpha \in [0,1]$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$.

**4.** *Monotonicity:* $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ if, and only if, $\alpha \geq \beta$.

Note that monotonicity implies that $a_1 \succ a_n$, which rules out the case $a_1 \sim \ldots \sim a_n$. Also, notice that monotonicity implies that for each gamble $g$ the probability $\alpha$ in the definition of continuity is uniquely defined[1].

**5.** *Substitution:* if $g = (p_1 \circ g^1, \ldots, p_k \circ g^k)$ and $h = (p_1 \circ h^1, \ldots, p_k \circ h^k)$, where $g_1, \ldots, h^k \in \mathcal{G}$, then $g^i \sim h^i$ for $i = 1, \ldots, k$ implies that $g \sim h$.

**6.** *Reduction to Simple Gambles:* for all $g \in \mathcal{G}$, $g \sim g_s$, the simple gamble associated to $g$.

As in the consumer theory, the utility function $u : \mathcal{G} \to \mathbb{R}$ represents $\succsim$ if for all $g', g \in \mathcal{G}$, $u(g') \geq u(g)$ if, and only if, $g' \succsim g$. We say that $u : \mathcal{G} \to \mathbb{R}$ representing $\succsim$ has the *expected utility property* if for all $g \in \mathcal{G}$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i),$$

where $(p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple gamble induced by $g$. We say that $u(g)$ is the expected utility of the gamble $g$.

**Theorem 1.** *Suppose $\succsim$ on $G$ satisfies properties (axioms) 1 to 6 above. Then there exists a utility function $u$ representing $\succsim$ that has the expected utility property.*

Proof: For each $g \in \mathcal{G}$, let $u(g) \in [0,1]$ be such that $g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)$. This number exists by continuity. Moreover, by monotonicity, $u(g)$ is uniquely defined for every gamble $g$. This defines a utility function $u : \mathcal{G} \to \mathbb{R}$.

We claim that $u$ represents $\succsim$. Indeed, by transitivity, $g' \succsim g$ implies that

$$(u(g') \circ a_1, (1-u(g')) \circ a_n) \succsim (u(g) \circ a_1, (1-u(g)) \circ a_n), \tag{1}$$

which implies that $u(g') \geq u(g)$ by monotonicity. Now observe, by monotonicity, that $u(g') \geq u(g)$ implies that 1 holds. Thus, by transitivity, $g' \succsim g$, which proves the claim.

[1] Why is that?

We now prove that $u$ has the expected utility property. By reduction to simple gambles, we only need to prove this for simple gambles. Let then $g = (p_1 \circ a_1, \ldots, p_n \circ a_n)$ and notice that for each $i \in \{1, \ldots, n\}$, $a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n) \equiv q^i$. By substitution, we then have that $g' = (p_1 \circ q^1, \ldots, p_n \circ q^n) \sim g$. It is easy to see that

$$g'_s = \left( \left( \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_1, \left( 1 - \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_n \right).$$

Thus, by reduction and transitivity,

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n) \sim g'_s \sim \left( \left( \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_1, \left( 1 - \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_n \right).$$

Since $u(g)$ is uniquely defined, it must be that $u(g) = \sum_{i=1}^{n} p_i u(a_i)$, the desired result. $\square$

It is immediate from the proof above that the existence of an utility function $u$ representing $\succsim$ follows from properties 2 to 4. Since a binary relation is a preference relation if it can be represented by an utility function, we then have that the completeness of $\succsim$ follows from transitivity, continuity, and monotonicity. The fact that $u$ has the expected utility property is a consequence of properties 5 and 6 (together with transitivity).

A decision maker with preferences $\succsim$ over compound gambles is an expected utility maximizer if these preferences are represented by a utility function with the expected utility property and the decision maker chooses the lottery $g$ that maximizes his expected utility. From now on we work with expected utility maximizers (despite the example below).

**Example (Allais Paradox):** Suppose that $A = \{2.5, 0.5, 0\}$, where each unit is measured in millions of euros. Consider now the following lotteries:

$$\begin{aligned}
g_1 &= 0.5, \\
g_2 &= \left( \frac{1}{10} \circ 2.5, \frac{89}{100} \circ 0.5, \frac{1}{100} \circ 0 \right), \\
g_3 &= \left( \frac{11}{100} \circ 0.5, \frac{89}{100} \circ 0 \right), \\
g_4 &= \left( \frac{1}{10} \circ 2.5, \frac{9}{10} \circ 0 \right).
\end{aligned}$$

Most people (when asked) prefer $g_1$ over $g_2$ and $g_4$ over $g_3$. This is not compatible with expected utility maximization. Indeed, if $\succsim$ is represented by an utility function $u$ with the expected utility property, then $g_1 \succ g_2$ if, and only if,

$$u(0.5) > \frac{1}{10}u(2.5) + \frac{89}{100}u(0.5) + \frac{1}{100}u(0),$$

which is equivalent to

$$\frac{11}{100}u(0.5) + \frac{89}{100}u(0) > \frac{1}{10}u(2.5) + \frac{9}{10}u(0).$$

This, however, implies that $g_3 \succ g_4$. Most people would agree that $g_3$ preferred to $g_4$ is insane. Since we know that expected utility follows from properties 5 and 6, the relevant question is whether one takes these properties as having normative content or a positive content. Most people would go with the second alternative.

**Theorem 2.** *Suppose that $u$ represents $\succsim$ and has the expected utility property. Then $v$ with the expected utility property represents $\succsim$ if, and only if, there exists $\alpha \in \mathbb{R}$ and $\beta > 0$ such that $v(g) = \alpha + \beta u(g)$ for all $g \in \mathcal{G}$.*

Proof: Sufficiency is obvious. Indeed, if $v$ is such that $v(g) = \alpha + \beta u(g)$ for every gamble $g$ with $\alpha \in \mathbb{R}$ and $\beta > 0$, then $v$ has the expected utility property.

In order to prove necessity we just need to consider simple gambles. For this, let $A = \{a_1, \ldots, a_n\}$, with $a_1 \succsim \ldots \succsim a_n$, and consider the simple gamble $g = (p_1 \circ a_1, \ldots, p_n \circ a_n)$. Moreover, assume that $a_1 \succ a_n$, for otherwise necessity is immediate[2].

Since $u$ represent $\succsim$, we have that $u(a_1) \geq \ldots \geq u(a_n)$. So, for each $i \in \{1, \ldots, n\}$, there exists $\alpha_i \in [0, 1]$ such that $u(a_i) = \alpha_i u(a_1) + (1 - \alpha_i)u(a_n)$; $\alpha_i > 0$ since $a_1 \succ a_n$. Since $u$ has the expected utility property, $\alpha_i u(a_1) + (1 - \alpha_i)u(a_n) = u(g)$, where $g = (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n)$. Since $u$ represents $\succsim$, we then have that

$$a_i \sim (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n). \tag{2}$$

[2] You should check this.

Now observe that $v$ also has the expected utility property and represents $\succsim$. Thus, by 2, $v(a_i) = \alpha_i v(a_1) + (1-\alpha_i)v(a_n)$. Hence, for all $i \in \{1, \dots, n\}$ such that $a_i \succ a_n$, we have that

$$\frac{u(a_1) - u(a_i)}{u(a_i) - u(a_n)} = \frac{1-\alpha_i}{\alpha_i} = \frac{v(a_1) - v(a_i)}{v(a_i) - v(a_n)},$$

which implies that

$$[u(a_1) - u(a_i)][v(a_i) - v(a_n)] = [v(a_1) - v(a_i)][u(a_i) - u(a_n)]. \tag{3}$$

Since 3 is also satisfied when $a_i \sim a_n$ and $u(a_1) > u(a_n)$, we then have (omitting the algebra) that

$$\begin{aligned} v(a_i) &= \frac{u(a_1)v(a_n) - v(a_1)u(a_n)}{u(a_1) - u(a_n)} + \frac{v(a_1) - v(a_n)}{u(a_1) - u(a_n)} u(a_i) \\ &= \alpha + \beta u(a_i), \end{aligned}$$

where $\beta > 0$ since $v(a_1) > v(a_n)$.

Therefore, since both $v$ and $u$ have the expected utility property,

$$v(g) = \sum_{i=1}^{n} p_i v(a_i) = \sum_{i=1}^{n} p_i(\alpha + \beta u(a_i)) = \alpha + \beta \sum_{i=1}^{n} p_i u(a_i) = \alpha + \beta u(g).$$

$\square$

The restriction to finite outcome spaces is done for simplicity. We can extend Theorem 1 to the case where $A$ is infinite, at the cost of some technicalities. In most applications, the case of interest is when $A = \mathbb{R}_+$ and an element of $A$ represents wealth. In this case, we can represent a simple gamble on $A$ by a cdf $F$[3]. For any simple gamble $F$, the expected utility of $F$ is

$$u(F) = \int_0^\infty u(w)dF(w).$$

We refer to the function $u : \mathbb{R}_+ \to \mathbb{R}$ such that $u(w)$ is the agent's payoff from the lottery $F$ that gives $w$ with probability one as the agent's utility function over wealth. Monotonicity implies that $u$ is strictly increasing.

[3]Recall that a cdf in $\mathbb{R}_+$ is an increasing and right-continuous function $F : \mathbb{R}_+ \to [0, 1]$ such that $F(0) = 0$ and $F(+\infty) = 1$.

## 3. Attitudes to Risk

Here and in the next two sections, assume that $A = \mathbb{R}_+$. Consider an individual with utility over wealth $u$. Now let $g$ be a gamble (over final wealth) and denote its expected value by $\mathbb{E}[g]$. We say the individual is: *(i)* risk–neutral at $g$ if $u(g) = u(\mathbb{E}[g])$; *(ii)* risk–averse at $g$ if $u(g) < u(\mathbb{E}[g])$; *(iii)* risk–loving at $g$ if $u(g) > u(\mathbb{E}[g])$.

We say the individual is

*(i) risk–neutral* if $u(g) = u(\mathbb{E}[g])$ for all gambles $g$;

*(ii) risk–averse* if $u(g) \leq u(\mathbb{E}[g])$ for all gambles $g$ and $u(g) < u(\mathbb{E}[g])$ for at least one non–degenerate gamble[4] $g$;

*(iii) risk–loving* if $u(g) \geq u(\mathbb{E}[g])$ for all gambles $g$ and $u(g) > u(\mathbb{E}[g])$ for at least one non–degenerate gamble $g$.

It is easy to see that if $u$ is linear (affine) in wealth, i.e., if $u(w) = \alpha + \beta w$ with $\alpha \in \mathbb{R}$ and $\beta > 0$, then the individual is risk-neutral. Now, for each $w, w' \in A$ and $t \in (0,1)$, let $g = (t \circ w, (1-t) \circ w')$. If the individual is risk–averse at $g$, then

$$u(tw + (1-t)w') \geq tu(w) + (1-t)u(w').$$

Thus, if the individual is risk–averse, his utility over wealth is concave and non–linear. It follows immediately from this that if the individual is risk–loving, his utility over wealth is convex and non–linear. Since a function can be concave and convex at the same time only if it is linear, we then have that if the individual is risk–neutral, his utility over wealth is linear. In other words, the individual is risk-neutral if, and only if, $u$ is linear.

The next result is useful.

**Lemma 1** (Jensen's Inequality)**.** *Suppose $u : \mathbb{R}_+ \to \mathbb{R}$ is concave. Then, for every cdf $F$,*

$$\int_0^\infty u(w)dF(w) \leq u\left(\int_0^\infty wdF(w)\right) = u(\mathbb{E}[F]).$$

[4] A gamble is non-degenerate if it assigns positive probability to some non-empty interval in $\mathbb{R}_+$.

Notice that Jensen's inequality reduces to

$$u(g) = \sum_{i=1}^{n} p_i u(g_i) \leq u\left(\sum_{i=1}^{n} p_i w_i\right) = u\left(\mathbb{E}[g]\right) \tag{4}$$

when $F$ is the cdf associated to the simple gamble $g = (p_1 \circ w_1, \ldots, p_n \circ w_n)$.

**Proof of Jensen's:** We establish Jensen's inequality in the case where $F$ assigns positive probability only to a finite subset of $\mathbb{R}_+$, in which case it can be represented by a simple gamble $g = (p_1 \circ w_1, \ldots, p_n \circ w_n)$. Since $u$ is concave, (4) is satisfied when $n = 2$. Suppose then, by induction, that (4) is satisfied when $n = K$, with $K \geq 2$. If $p_{K+1} \in \{0, 1\}$, (4) is satisfied when $n = K + 1$. So assume that $p_{K+1} \notin \{0, 1\}$ and observe that

$$\sum_{i=1}^{K+1} p_i w_i = \sum_{i=1}^{K} p_i \overline{w} + p_{K+1} w_{K+1} = (1 - p_{K+1})\overline{w} + p_{K+1} w_{K+1},$$

where $\overline{w} = \sum_{i=1}^{K} p_i w_i / \sum_{i=1}^{K} p_i$; notice that $\overline{w}$ is well–defined since $\sum_{i=1}^{K} p_i = 1 - p_{K+1} > 0$. Now observe that

$$\begin{aligned} u\left(\sum_{i=1}^{K+1} p_i w_i\right) &\geq (1 - p_{K+1}) u(\overline{w}) + p_{K+1} u(w_{K+1}) \\ &\geq \sum_{i=1}^{K} p_i w_i + p_{K+1} w_{K+1}, \end{aligned}$$

where the first inequality follows from the concavity of $u$ and the second inequality follows from the induction hypothesis (why?). Hence, (4) holds when $n = K + 1$. By induction, (4) holds for all $n \geq 2$. $\square$

It is easy to see, by Jensen's inequality, that if $u$ is concave, then $u(g) \leq u\left(\mathbb{E}[g]\right)$ for every gamble $g$. Now observe that if $u$ is non–linear, there exists $w, w' \in \mathbb{R}_+$ with $w \neq w'$ and $t \in (0, 1)$ such that $u(tw + (1-t)w') > tu(w) + (1-t)u(w')$. Let then $g = (t \circ w, (1-t) \circ w')$; notice that $g$ is a non–degenerate gamble. By construction, $u(g) < u\left(\mathbb{E}[g]\right)$. Thus, if $u$ is concave and non–linear, the individual is risk–averse. Therefore, from the discussion before Jensen's inequality, the individual is risk–averse if, and only if, his utility over wealth is concave and non–linear. It is the immediate to see from this that the individual is risk–loving if, and only if, his utility over wealth is convex and non–linear.

**Example:** Consider and individual with utility over wealth $u$, where $u$ is strictly increasing and strictly concave; thus, the individual is risk–averse. Let $w_0 > 0$ be the individual's initial wealth. There is a probability $\alpha \in (0, 1)$ that he loses an amount $0 < L < w_0$ of his wealth. The individual can insure himself against this loss. The price of one unit of insurance is $\rho$. We assume that insurance is actuarially fair, i.e., $\rho = \alpha$.

The individual's expected payoff from contracting an amount $x$ of insurance is

$$\begin{aligned} v(x) &= \alpha u(w_0 - L + x - \rho x) + (1 - \alpha)u(w_0 - \rho x) \\ &= \alpha u(w_0 - L + x - \alpha x) + (1 - \alpha)u(w_0 - \alpha x). \end{aligned}$$

Indeed, with probability $1 - \alpha$ he suffers no loss, and so his final wealth is $w_0 - \alpha x$. With probability $\alpha$ he suffers a loss, and so his final wealth is $w_0 - L + x - \alpha x$; the individual pays for insurance regardless of whether he suffers a loss of not. The problem of the individual is to choose the amount of insurance $x \in [0, w_0/\alpha]$ that maximizes $v(x)$. It is easy to show that $v$ is strictly concave, so that the solution to the individual's problem is characterized by its first–order condition. In particular, if there exists $x^* \in (0, w_0/\alpha)$ such that $v'(x^*) = 0$, then $x^*$ is the solution. Notice that

$$v'(x) = (1 - \alpha)\alpha u'(w_0 - L + (1 - \alpha)x) - \alpha(1 - \alpha)u'(w_0 - \alpha x).$$

Since $u'$ is strictly decreasing, $v'(x^*) = 0$ if, and only if,

$$w_0 - L + (1 - \alpha)x^* = w_0 - \alpha x^* \Leftrightarrow x^* = L.$$

Thus, it is optimal for the individual to insure himself completely against the loss.

## 4. Risk Aversion

### Measuring the Risk of Gambles

Consider an individual with a concave utility function over wealth. For any lottery $g$, the certainty equivalent $CE$ of $g$ is the amount of wealth such that $u(g) = u(CE)$. Since $u$ is strictly increasing and concave, Jensen's inequality implies that $CE \leq \mathbb{E}[g]$ for every gamble $g$. The risk premium of $g$ is the value $P$ such that $u(g) = u(\mathbb{E}[g] - P)$; i.e., $P = \mathbb{E}[g] - CE$.

The risk premium of a gamble measures how much the (risk–averse) individual is willing to pay in order to avoid the risk associated with the gamble.

**The Arrow-Pratt Measure of Absolute Risk Aversion**

Suppose that $u$ is twice differentiable with $u' > 0$; since $u$ is concave, $u'' \leq 0$. The Arrow-Pratt measure of absolute risk aversion is

$$R_a(w) = -\frac{u''(w)}{u'(w)}.$$

Consider two individuals, 1 and 2, with utilities over wealth $u$ and $v$, respectively, and suppose that for all $w \geq 0$,

$$R_a^1(w) = -\frac{u''(w)}{u'(w)} > R_a^2(w) = -\frac{v''(w)}{v'(w)}. \tag{5}$$

For simplicity assume that $v(w) \geq 0$ for all $w \geq 0$. Hence, we can define $h : [0, +\infty) \to \mathbb{R}$ to be such that $h(x) = u(v^{-1}(x))$. Notice that

$$h'(x) = \frac{u'(v^{-1}(x))}{v'(v^{-1}(x))} > 0$$

and that

$$\begin{aligned} h''(x) &\propto u''(v^{-1}(x)) - v''(v^{-1}(x))\frac{u'(v^{-1}(x))}{v'(v^{-1}(x))} \\ &\propto -R_a^1(w) + R_a^2(w) < 0 \end{aligned}$$

by 5. Therefore, $h$ is strictly increasing and strictly concave.

Now let $F$ be a cdf and define $\widehat{w}_1$ and $\widehat{w}_2$ be such that

$$\begin{aligned} u(\widehat{w}_1) &= \int_0^\infty u(w)dF(w), \\ v(\widehat{w}_2) &= \int_0^\infty v(w)dF(w); \end{aligned}$$

$\widehat{w}_1$ is the certainty equivalent for 1 of the gamble defined by $F$ and $\widehat{w}_2$ is the certainty equivalent for 2 of the same gamble. Since $h(v(w)) = u(w)$,

$$u(\widehat{w}_1) = \int_0^\infty h(v(w))dF(w) < h\left(\int_0^\infty v(w)dF(w)\right) = h(v(\widehat{w}_2)) = u(\widehat{w}_2),$$

where the inequality follows from Jensen's inequality. Thus, since $u$ is strictly increasing, $\widehat{w}_2 > \widehat{w}_1$. Consequently, for every gamble $g$, the risk premium of $g$ is greater for 1 than for 2. In other words, 1 is more risk–averse than 2. In particular, if 1 and 2 have the same wealth, then 2 accepts any gamble that 1 accepts, and 1 rejects any gamble than 2 rejects.

**Example (DARA):** We say an individual has decreasing absolute risk–aversion (DARA) if his utility over wealth is such that $R_a(w)$ is strictly decreasing in $w$.

Consider an individual with DARA who must decide how much of his initial wealth $w_0 > 0$ he allocates to a risky asset. The asset has $n$ possible rates of return, $r_1$ to $r_n$, with probability of $r_i$ equal to $p_i \in (0, 1)$.

Let $u$ be the individual's utility over wealth and assume $u'' < 0$. Now let $\beta$ be the amount of wealth he allocates to the risky asset. The individual's final wealth is then $w_0 - \beta + (1 + r_i)\beta = w_0 + \beta r_i$ with probability $p_i$. Thus, the problem of the individual is to choose $\beta \in [0, w_0]$ that maximizes

$$v(\beta) = \sum_{i=1}^{n} p_i u(w_0 + \beta r_i).$$

It is easy to see that $u'' < 0$ implies that $v'' < 0$. So, the solution to the individual's problem is unique and is characterized by its first-order condition. Denote by $\beta^*$ the optimal choice of $\beta$. Notice that

$$v'(\beta) = \sum_{i=1}^{n} p_i r_i u'(w_0 + \beta r_i).$$

Thus,

$$v'(0) = \sum_{i=1}^{n} p_i r_i u'(w_0) = u'(w_0)\mathbb{E}[r],$$

where $\mathbb{E}[r]$ is the expected return of the asset. Hence, $\mathbb{E}[r] > 0$ is sufficient for $\beta^* > 0$; in fact, $\mathbb{E}[r] > 0$ is also necessary for $\beta^* > 0$ (prove this). Now observe that

$$v'(w_0) = \sum_{i=1}^{n} p_i r_i u'((1 + r_i)w_0),$$

and that a necessary and sufficient condition for $\beta^* < w_0$ is that $v'(w_0) < 0$. A necessary condition for $v'(w) < 0$ is that $r_i < 0$ for at least one $i$. In what follows, assume that $\beta^*$ is interior.

By assumption, $\beta^*$ is the unique solution to $v'(\beta^*) = 0$. This equation defines $\beta^*$ implicitly as a function of $w$ (and of the rates of return $r_i$ as well). By the Implicit Function Theorem, we have that $\beta^*$ is differentiable and that

$$\frac{d\beta^*}{dw} = \frac{-\sum_{i=1}^n p_i r_i u''(w + \beta^* r_i)}{\sum_{i=1}^n p_i r_i^2 u''(w + \beta^* r_i)}.$$

The denominator of $d\beta^*/dw$ is negative. Thus $d\beta^*/dw > 0$, i.e., the risky asset is a normal good, if the numerator is negative as well. Now notice that

$$-\sum_{i=1}^n p_i r_i u''(w + \beta^* r_i) = \sum_{i=1}^n p_i u'(w + \beta^* r_i) R_a(w + \beta^* r_i) r_i.$$

Since $R_a(w)$ is strictly decreasing in $w$, $r_i > 0$ implies that $R_a(w)r_i > R_a(w + \beta^* r_i)r_i$ and $r_i < 0$ implies that $R_a(w)r_i > R_a(w + \beta^* r_i)r_i$. Thus, since at least one $r_i$ is different from zero (otherwise $\mathbb{E}[r] = 0$), we have that

$$-\sum_{i=1}^n p_i r_i u''(w + \beta^* r_i) < \sum_{i=1}^n p_i r_i u'(w + \beta^* r_i) R_a(w) = 0,$$

by the first-order condition. Hence, DARA implies the risky asset is normal.

## 5. Stochastic Dominance

First, let us understand better what the integral

$$\int_0^\infty u(w) dF(w) \tag{6}$$

means. If the cdf $F$ has a density $f$, i.e., if there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F(w) = \int_0^w f(s) ds$$

for all $w \geq 0$, then

$$\int_0^\infty u(w) dF(w) = \int_0^\infty u(w) f(w) dw.$$

If $F$ is the cdf associated with the simple lottery $(p_1 \circ w_1, \ldots, p_n \circ w_n)$, then

$$\int_0^\infty u(w) dF(w) = \sum_{i=1}^n p_i u(w_i) = \sum_{i=1}^n [F(w_i) - F(w_{i-1})] u(w_i),$$

where we adopt the convention that $F(w_0) = 0$.

The integral (6) is well–defined for any cdf $F$, whether it has a density of not. It has the property that if $u = \mathbb{I}_{(a,b]}$, the characteristic function of the interval $(a, b]$, then

$$\int_0^\infty u(w)dF(w) = F(b) - F(a) = Pr\{a < w \leq b\}. \tag{7}$$

In fact, property (7) completely determines the integral (6) in the following sense. For any two cdfs $F$ and $G$, write $F \equiv G$ if $F(b) - F(a) = G(b) - G(a)$ for all $0 \leq a < b < \infty$. Then, $F \equiv G$ implies that

$$\int_0^\infty u(w)dF(w) = \int_0^\infty u(w)dG(w)$$

for every (integrable) function $u : \mathbb{R}_+ \to \mathbb{R}$.

In what follows we refer to the probability distribution on $\mathbb{R}_+$ induced by the cdf $F$ as (the distribution) $F$. The integral (6) is the expected value of the random variable $u$ given the distribution $F$.

**First-Order Stochastic Dominance**

The distribution $F$ first-order stochastically dominates the distribution $G$, $F \succsim_{fosd} G$, if for all increasing $u : \mathbb{R}_+ \to \mathbb{R}$, we have that

$$\int_0^\infty u(w)dF(w) \geq \int_0^\infty u(w)dG(w).$$

Then, given two (simple) lotteries $F$ and $G$, any expected utility maximizer prefers $F$ over $G$ if $F \succsim_{fosd} G$. It is possible to show that if $F \sim_{fosd} G$, then $F \equiv G$.

**Lemma 2.** *$F \succsim_{fosd} G$ if, and only if, $F(w) \leq G(w)$ for all $w \geq 0$.*

**Proof:** *Necessity.* For each $w \geq 0$, let $u = \mathbb{I}_{(w,+\infty)}$. Then $u$ is increasing, and so

$$1 - F(w) = \int_0^\infty u(w)dF(w) \geq \int_0^\infty u(w)dG(w) = 1 - G(w),$$

which implies the desired result.

*Sufficiency.* We prove sufficiency for discrete cdfs. Let $F$ be the cdf associated to the gamble $(p_1 \circ w_1, \ldots, p_n \circ w_n)$ and $G$ be the cdf associated to the gamble $(q_1 \circ w'_1, \ldots, q_m \circ w'_m)$.

Without loss, we can assume that $m = n$ and $w_i = w_i'$ for all $i \in \{1, \ldots, n\}$. Moreover, let $w_1 \leq \ldots \leq w_n$.

Now observe that if $H$ is the cdf associated to $(r_1 \circ w_1, \ldots, r_n \circ w_n)$, then

$$\begin{aligned}\sum_{i=1}^{n} r_i u(w_i) &= \sum_{i=1}^{n}[H(w_i) - H(w_{i-1})]u(w_i) \\ &= u(w_n) + \sum_{i=1}^{n-1} H(w_i)u(w_i) - \sum_{i=2}^{n} H(w_{i-1})u(w_i) \\ &= u(w_n) - \sum_{i=1}^{n-1} H(w_i)[u(w_{i+1}) - u(w_i)].\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} p_i u(w_i) - \sum_{i=1}^{n} q_i u(w_i) = \sum_{i=1}^{n-1} \underbrace{[G(w_i) - F(w_i)]}_{\geq 0} \underbrace{[u(w_{i+1}) - u(w_i)]}_{\geq 0} \geq 0.$$

$\square$

An immediate corollary of Lemma 2 is that $F \succsim_{fosd} G$ implies that $\mathbb{E}[F] \geq \mathbb{E}[G]$. The converse is not true.

**Example (Upward Probabilistic Shift):** Consider the following two-stage lottery. First, draw $w$ according to the cdf $G$. Once $w$ is drawn, draw $z$ according to the cdf $H_w$ with $H_w(0) = 0$. The final wealth is $w' = w + z$. Denote the distribution implied by the two-stage lottery by $F$. The cdf of $F$ is

$$F(w') = Pr\{w + z \leq w'\} = Pr\{z \leq w' - w\} = \int_0^\infty H_w(w' - w)dG(w)$$

and we say that $F$ is an upward probabilistic shift of $G$. Since $H_w(z) = 0$ if $z \leq 0$ and $H_w(z) \leq 1$ for all $z$, we have that

$$F(w') \leq \int_0^{w'} H_w(w' - w)dG(w) \leq \int_0^{w'} dG(w) = G(w').$$

Thus, $F \succsim_{fosd} G$. It is possible to show (but quite difficult) that if $F \succsim_{fosd} G$, then $F$ is an upward probabilistic shift of $G$.

**Second-Order Stochastic Dominance**

The distribution $F$ second-order stochastically dominates $G$, $F \succsim_{sosd} G$, if for all concave $u : \mathbb{R}_+ \to \mathbb{R}$,

$$\int_0^\infty u(w)dF(w) \geq \int_0^\infty u(w)dG(w).$$

Then, given two (simple) lotteries $F$ and $G$, any risk–averse expected utility maximizer weakly prefers $F$ over $G$ if $F \succsim_{sosd} G$. Notice that if $F \succsim_{sosd} G$, then $Var(F) \leq Var(G)$. The converse is not true, though[5].

**Example (Mean–Preserving Spread):** Consider the following two-stage lottery. First, draw $w$ according to $G$. Once $w$ is drawn, obtain $z$ from $H_w$ (defined on $\mathbb{R}$) such that $\mathbb{E}[H_w] = 0$ and $H_w(-w) = 0$ for all $w$. The final wealth is $w' = w + z$. Denote the distribution implied by the two-stage lottery by $F$. The cdf of $F$ is

$$F(w') = Pr\{w + z \leq w'\} = Pr\{z \leq w' - w\} = \int_0^\infty H_w(w' - w)dG(w).$$

We say that $F$ is a mean-preserving spread of $G$.

In what follows, assume $G$ has a density and $H_w$ has a density for all $w$, and denote the densities of $G$ and $H_w$ by $g$ and $h_w$, respectively. Then

$$\begin{aligned}
F(w') &= \int_0^\infty H_w(w' - w)g(w)dw \\
&= \int_0^\infty \left( \int_{-w}^{w'-w} h_w(s)ds \right) g(w)dw \\
&= \int_{-w}^{w'-w} \left( \int_0^\infty h_w(s)g(w)dw \right) ds \\
&= \int_0^{w'} \left( \int_0^\infty h_w(s - w)g(w)dw \right) ds
\end{aligned}$$

where the second equality follows from the fact that $H_w(-w) = 0$, the third equality follows from Fubini's theorem, and the last equality follows from the change of variable $s \mapsto s - w$. Thus, $F$ has a density, which is given by

$$f(s) = \int_0^\infty h_w(s - w)g(w)dw.$$

[5] See the papers by Rothschild and Stiglitz for a more detailed explanation of why one shouldn't compare the risk of two distributions by their variances.

Hence,

$$\begin{aligned}\int_0^\infty u(w')dF(w') &= \int_0^\infty u(w')f(w')dw' \\ &= \int_0^\infty \left(\int_0^\infty h_w(w'-w)g(w)dw\right)u(w')dw' \\ &= \int_0^\infty \left(\int_0^\infty u(w')h_w(w'-w)dw'\right)g(w)dw \\ &= \int_0^\infty \left(\int_{-w}^\infty u(w+z)h_w(z)dz\right)g(w)dw \\ &= \int_0^\infty \left(\int_{-w}^\infty u(w+z)dH_w(z)\right)dG(w),\end{aligned}$$

where the third equality follows from Fubini and the fourth equality follows from the change of variable $w' \mapsto z + w$.

First, notice that

$$\int_0^\infty w'dF(w') = \int_0^\infty \left(\int_{-w}^\infty (w+z)dH_w(z)\right)dG(w) = \int_0^\infty wdG(w).$$

Thus (confirming intuition), $\mathbb{E}[F] = \mathbb{E}[G]$. Now observe that if $u$ is concave, then Jensen's inequality implies that

$$\begin{aligned}\int_0^\infty \left(\int_{-w}^\infty u(w+z)dH_w(z)\right)dG(w) &\leq \int_0^\infty u\left(\int_{-w}^\infty (w+z)dH_w(z)\right)dG(w) \\ &= \int_0^\infty u(w)dG(w).\end{aligned}$$

Thus, $G \succsim_{sosd} F$. The converse is also true (but also quite difficult to prove): if $G \succsim_{sosd} F$, then $F$ is a mean-preserving spread of $G$.