General Equilibrium

1. Exchange Economies

We assume there is a finite number n of goods and the consumption set of each agent is $\mathbb{R}^n_+$. In what follows, we use superscripts to label agents and subscripts to label goods. So, for example, $x^i \in \mathbb{R}^n_+$ denotes a consumption vector for agent i and x^i_k denotes the amount of good k that agent i consumes.

An *exchange economy* is a list $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$, where:

(i) $\mathcal{I} = \{1, \dots, I\}$ is a finite set of agents;

(ii) $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ is agent i's utility function;

(iii) $e^i \in \mathbb{R}^n_+$ is agent i's endowment.

We always assume that $e = \sum_{i \in \mathcal{I}} e^i \gg 0$, where e is the aggregate endowment vector. By construction e_k is the total amount of good k in the economy. So, $e \gg 0$ implies there is a positive amount of each good in the economy.

Consider an exchange economy $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$. An *allocation* is a list $x = (x^1, \dots, x^I)$, where $x^i \in \mathbb{R}^n_+$ is a consumption vector for agent i. An allocation $x = (x^1, \dots, x^I)$ is *feasible* if $\sum_{i \in \mathcal{I}} x^i \leq \sum_{i \in \mathcal{I}} e^i$. In words, an allocation is feasible if the total consumption of each good is no greater than its total supply; disposal of goods is not allowed. Note that the set of feasible allocations only depends on the aggregate endowment vector $e = \sum_{i \in \mathcal{I}} e^i$. We denote the set of feasible allocations by $F(e)$.

If $x, \overline{x}$ are two feasible allocations such that $u^i(x^i) \geq u^i(\overline{x}^i)$ for all $i \in \mathcal{I}$ and $u^i(x^i) > u^i(\overline{x}^i)$ for at least one $i \in \mathcal{I}$, we say that x *Pareto dominates* $\overline{x}$. A feasible allocation $\overline{x}$ is *Pareto efficient/optimal* if there exists no feasible allocation x that Pareto dominates $\overline{x}$. We refer to the set of Pareto efficient allocations in an exchange economy as the *contract curve.*

Definition 1. *Let $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$ be an exchange economy. A pair (p^*, x^*), where $p^* \gg 0$ is a vector of prices and x^* is an allocation, is a* Walrasian equilibrium *if:*

*(*i*) for all $i \in \mathcal{I}$, x^{*i} solves*

$$\begin{array}{ll} \max & u^i(x) \\ s.t. & p^* \cdot x \leq p^* \cdot e^i \ ; \\ & x \in \mathbb{R}^n_+ \end{array}$$

*(*ii*)* $\sum_{i \in \mathcal{I}} x^{*i} = \sum_{i \in \mathcal{I}} e^i$.

In words, a pair (p^*, x^*) is a Walrasian equilibrium if x^* is feasible and each agent i maximizes his utility given the vector of prices p^*. We say an allocation x^* is a *Walrasian allocation* if there exists $p^* \gg 0$ such that (p^*, x^*) is a Walrasian equilibrium. The set of Walrasian allocations depends on the initial allocation $(e^1, \ldots, e^I)$ and on the preferences of each agent. We denote the set of Walrasian allocations by $W(\mathcal{E})$.

There are two important behavioral assumptions in the definition of a Walrasian equilibrium. The first, the so-called *competitive hypothesis*, is that agents take prices as given. In other words, agents believe that their decisions do not have an impact on prices. The second assumption is that given a vector of prices, agents believe that they can buy and sell as much as they want of each good, as long as this is possible given their budget constraint. Also note that the Walrasian theory of markets has nothing to say about the formation of prices.

2. Existence

The first question we address is the existence of Walrasian equilibria.

Consider an exchange economy $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$ where the utility functions u^i are continuous and strictly quasi-concave. By assumption, for every vector of prices $p \gg 0$ and every wealth $w \geq 0$, the consumer's problem

$$\begin{array}{ll} \max & u^i(x) \\ s.t. & p \cdot x \leq w \\ & x \in \mathbb{R}^n_+ \end{array}$$

for agent i has a unique solution. Denote this solution by $x^i(p, w)$. Now let

$$z_k(p) = \sum_{i \in \mathcal{I}} \left[x^i_k(p, p \cdot e^i) - e^i_k \right] .$$

By construction, $\sum_{i \in \mathcal{I}} x_k^i(p, p \cdot e^i)$ is the aggregate demand of good k when the vector of prices is p. Thus, $z_k(p)$ is the aggregate excess demand of good k when the vector of prices is p. To finish, let $z : \mathbb{R}^n_{++} \to \mathbb{R}$ be given by $z(p) = (z_1(p), \ldots, z_n(p))$. Notice that

$$z(p) = \sum_{i \in \mathcal{I}} \left[x^i(p, p \cdot e^i) - e^i \right].$$

The following result is straightforward.

Lemma 1. *The pair (p^*, x^*) is a Walrasian equilibrium if, and only if, $x^{*i} = x^i(p^*, p^* \cdot e^i)$ for all $i \in \mathcal{I}$ and $z(p^*) = 0$.*

Proof: Suppose (p^*, x^*) is a Walrasian equilibrium. By assumption, $x^{*i} = x^i(p^*, p^* \cdot e^i)$ for all $i \in \mathcal{I}$ and

$$\sum_{i \in \mathcal{I}} x^{*i} = \sum_{i \in \mathcal{I}} e^i \Leftrightarrow \sum_{i \in \mathcal{I}} \left[x^{*i} - e^i \right] = z(p^*) = 0.$$

Suppose now that (p^*, x^*) is such that $x^{*i} = x^i(p^*, p^* \cdot e^i)$ for all $i \in \mathcal{I}$ and $z(p^*) = 0$. It is immediate to see that (p^*, x^*) is a Walrasian equilibrium. $\square$

The above result shows that the exchange economy $\mathcal{E}$ has a Walrasian equilibrium if, and only if, there exists $p^* \gg 0$ such that $z(p**) = 0$. Thus, if we can determine conditions under which the equation $z(p) = 0$ has a solution in $\mathbb{R}^n_{++}$, we have conditions under which $\mathcal{E}$ has a Walrasian equilibrium. The following result is useful.

Lemma 2. *The function z has the following properties:*

*(*i*) z is continuous;*

*(*ii*) z is homogenous of degree zero.*

Suppose the utility functions u^i are also locally non–satiated. Then:

*(*iii*) $p \cdot z(p) = 0$ for all $p \gg 0$.*

Suppose now the utility functions u^i are strongly increasing. Then:

*(*iv*) if $\{p^m\}$ is a sequence of prices in $\mathbb{R}^n_{++}$ that converges to $p \in \partial\mathbb{R}^n_{++} \backslash \{0\}$, then there exists $k \in \{1, \ldots, n\}$ with $p_k = 0$ such that the sequence $\{z_k(p^m)\}$ is not bounded above.*

Property (ii) is intuitive. Multiplying the vector of prices p by a positive constant does not change relative prices, and so does not change the agents' demands. Thus, if (p^*, x^*) is a Walrasian equilibrium, then $(\alpha p^*, x^*)$ is also a Walrasian equilibrium for all $\alpha > 0$. There are different ways of eliminating this uninteresting multiplicity of equilibria. One is to normalize prices so that they sum up to one. Another is to set the price of one of the goods to one[1].

Property (iv) is also intuitive. To see why, consider what happens when the price of one of the goods, let us say good 1, converges to zero. Since $\sum_i e^i \gg 0$, there exists at least one agent whose wealth remains bounded away from zero. Since utility functions are strongly increasing, this agent can increase his utility by increasing the consumption of good 1 as its price decreases. In the limit, the demand for good 1 is infinite. Notice that if the price of two or more goods converge to zero, it need not be the case that the demand for *all* of those goods increases without bound, as relative price matters. What (iv) says is that the demand for at least one of the goods with zero price in the limit must increase without bound.

Property (iii) is the so–called Walras' law. We say the market for good k clears at the price $p \gg 0$ if $z_k(p) = 0$. Walras' law then implies that if the market for $n-1$ goods clear at the price p, then the market for the remaining good also clears at the price p.

Proof of Lemma 2: The continuity of z follows from the joint continuity of the demand functions $x^i(p, w)$ on p and w. Since the budget set of agent i's consumer problem with $w = p \cdot e_i$ does not change when we multiply all prices by the same positive constant, we have that $w^i(p, p \cdot e^i) = x^i(\alpha p, \alpha p \cdot e^i)$ for all $\alpha > 0$. Thus, $z(\alpha p) = z(p)$ for all $\alpha > 0$, i.e., z is homogenous of degree zero. We now prove (iii). For this, notice that u^i locally non–satiated

[1] We usually refer to this good as the *numeraire*.

implies that $p \cdot x^i(p, p \cdot e^i) = p \cdot e^i$ for all $i \in \mathcal{I}$. Thus,

$$\begin{aligned}
p \cdot z(p) &= \sum_{k=1}^{n} p_k z_k(p) \\
&= \sum_{k=1}^{n} \sum_{i=1}^{I} p_k \left[x_k^i(p, p \cdot e^i) - e_k^i \right] \\
&= \sum_{i=1}^{I} \sum_{k=1}^{n} p_k \left[x_k^i(p, p \cdot e^i) - e_k^i \right] \\
&= \sum_{i=1}^{I} \left[p \cdot x^i(p, p \cdot e^i) - p \cdot e^i \right] \\
&= 0.
\end{aligned}$$

The proof of property *(iv)* is in Chapter 5 of Jehle & Reny. □

We present the next result without proof[2].

Theorem 1. *Let $z : \mathbb{R}^n_{++} \to \mathbb{R}$ be a function satisfying properties (*i*) to (*iv*) in Lemma 2. Then there exists $p^* \gg 0$ such that $z(p^*) = 0$.*

The following result is an immediate corollary of Lemma 1 and Theorem 1. It establishes that under certain conditions, the set $W(\mathcal{E})$ of an exchange economy $\mathcal{E}$ is not empty.

Corollary 1. *Let $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$ with $\sum_{i \in \mathcal{I}} e^i \gg 0$ be such that the functions u^i are continuous, strictly quasi–concave, and strongly increasing. Then $\mathcal{E}$ has a Walrasian equilibrium.*

3. First and Second Fundamental Welfare Theorems

Now, we introduce a notion of equilibrium that allows for more generality on the distribution of wealth among consumers than the (private) endowments we have used so far. In other words, we are considering the idea that a social planner can unilaterally redistribute the consumers' wealth using lump-sum transfers in any way the social planner desires.

Definition 2. *Given an exchange economy $\mathcal{E} = \{u^i, e^i\}_{i \in \mathcal{I}}$, an allocation x^* and a price vector $p^* = (p_1, \ldots, p_n)$ constitute a price equilibrium with transfers if there is an assignment of wealth levels $(w^1, \ldots, w^I)$ with $\sum_{i \in \mathcal{I}} w^i = p^* \cdot e$ such that*

[2] Chapter 5 of Jehle & Reny contains a proof.

*(*i*) for all* $i \in \mathcal{I}$*,* x^{*i} *solves*

$$\begin{aligned} \max \quad & u^i(x) \\ s.t. \quad & p^* \cdot x \leq w^i \;; \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

*(*ii*)* $\sum_{i\in\mathcal{I}} x^{*i} = \sum_{i\in\mathcal{I}} e^i$.

We can clearly see that a Walrasian equilibrium is a particular case of a price equilibrium with transfers, in which the redistribution made by the social planner is null.

Theorem 2 (First Fundamental Welfare Theorem)**.** *Consider an exchange economy* $\mathcal{E} = \{u^i, e^i\}_{i\in\mathcal{I}}$. *If the utility functions* u^i *are locally non–satiated, and if* (p^*, x^*) *is a price equilibrium with transfers, then the allocation* x^* *is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.*

Proof: Suppose that (p^*, x^*) is a price equilibrium with transfers and that the associated wealth levels are $(w^1, \ldots, w^I)$.

The preference maximization part of the definition of a price equilibrium with transfers implies that, if $u^i(x^i) > u^i(x^{i*})$ then $p^* \cdot x^i > w^i$, which together with local non-satiation of the utility functions implies the additional property that if $u^i(x_i) \geq u^i(x^{i*})$, then $p^* \cdot x^i \geq w^i$.

Now, consider an allocation x^i that Pareto dominates x^{i*}, i.e. $u^i(x^i) \geq u^i(x^{i*})$ for all $i \in \mathcal{I}$ and $u^i(x^i) > u^i(x^{i*})$ for some $i \in \mathcal{I}$. Then, by the argument in the last paragraph, we must have $p^* \cdot x^i \geq w^i$ for all $i \in \mathcal{I}$, and $p^* \cdot x^i > w^i$ for some i, and hence $\sum_{i\in\mathcal{I}} p^* \cdot x^i > \sum_{i\in\mathcal{I}} w^i = \sum_{i\in\mathcal{I}} p^* \cdot e^i$, contradicting feasibility. $\square$

Loosely speaking, the First Fundamental Welfare Theorem (FFWT) implies that any Walrasian allocation exhausts all gains from trade. The Second Fundamental Welfare Theorem, which we prove next, is a partial converse of the FFWT. We start with the following result, which is interesting by itself.

Theorem 3. *Let* $\mathcal{E} = \{u^i, e^i\}_{i\in\mathcal{I}}$ *with* $\sum_{i\in\mathcal{I}} e^i \gg 0$ *be an exchange economy where the functions* u^i *are continuous, strictly quasi–concave, and strongly increasing. If the initial allocation* $\widetilde{e} = (e^1, \ldots, e^I)$ *is Pareto efficient, then* $\widetilde{e}$ *is the unique Walrasian equilibrium.*

Proof: We know by Corollary 1 that a Walrasian equilibrium (p^*, x^*) exists. Suppose, by contradiction, that $x^* \neq \widetilde{e}$. We know that $u^i(x^i) \geq u^i(e^i)$ for all $i \in \mathcal{I}$, since e^i is an element of agent i's budget set regardless of the vector of prices. Since $\widetilde{e}$ is Pareto efficient, it must then be that $u^i(x^{*i}) = u^i(e^i)$ for all $i \in \mathcal{I}$. Now, let $\widehat{x} = (\widehat{x}^1, \ldots, \widehat{x}^I)$ be such that $\widehat{x}^i = \lambda x^{*i} + (1-\lambda)e^i$ for all $i \in \mathcal{I}$, where $\lambda \in (0,1)$. Notice that

$$\begin{aligned}\sum_{i\in\mathcal{I}} \widehat{x}^i &= \lambda \sum_{i\in\mathcal{I}} x^{*i} + (1-\lambda)\sum_{i\in\mathcal{I}} e^i \\ &= \lambda \sum_{i\in\mathcal{I}} e^i + (1-\lambda)\sum_{i\in\mathcal{I}} e^i \\ &= \sum_{i\in\mathcal{I}} e^i,\end{aligned}$$

which implies that $\widehat{x}$ is feasible. Since each u^i is strictly quasi-concave, we then have that $u^i(\widehat{x}^i) \geq \min\{u^i(x^{*i}), u^i(e^i)\} = u^i(e^i)$ for all $i \in \mathcal{I}$, where $u^i(\widehat{x}^i) > u^i(e^i)$ if $x^{*i} \neq e^i$. Since $x^* \neq \widetilde{e}$ by assumption, it must be that $x^{*i} \neq e^i$ for at least one $i \in \mathcal{I}$. Hence, $\widehat{x}$ Pareto dominates $\widetilde{e}$, a contradiction. Thus, $x^* = \widetilde{e}$, and so $\widetilde{e}$ is the unique Walrasian allocation. $\square$

Theorem 3 tells us that under certain conditions, if the initial allocation in an exchange economy is Pareto efficient, then the initial allocation is the unique Walrasian allocation. A natural question to ask is if in general it is the case that exchange economies have a unique Walrasian allocation. The answer is no.

The Second Fundamental Welfare Theorem (SFWT) is an immediate consequence of the above result. The SFWT implies that any Pareto efficient allocation is a Walrasian allocation of an exchange economy we obtain after a suitable redistribution of the agents' endowment. The SFWT is an useful result since determining the set of Pareto efficient allocations is simpler than determining the set of Walrasian allocations.

Theorem 4 (Second Fundamental Welfare Theorem)**.** *Let* $\mathcal{E} = \{u^i, e^i\}_{i\in\mathcal{I}}$ *with* $\sum_{i\in\mathcal{I}} e^i \gg 0$ *be an exchange economy where the functions* u^i *are continuous, strictly quasi–concave, and strongly increasing. Suppose that* $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^I)$ *is a Pareto efficient allocation. If we redistribute the initial endowments in* $\mathcal{E}$ *in such a way that* $\overline{x}$ *is the new vector of endowments, then* $\overline{x}$ *in a Walrasian allocation in* $\mathcal{E}' = \{u^i, \overline{x}^i\}_{i\in\mathcal{I}}$.

We can relax the assumption of strict quasi–concave utility functions in the SFWT if we restrict attention to interior Pareto efficient allocations - an allocation x is interior if $x^i \gg 0$ for all $i \in \mathcal{I}$. The proof relies on the Separating Hyperplane Theorem[3].

4. Production Economies

An *economy with production* is a list $\mathcal{E} = (\{u^i, e^i\}_{i \in \mathcal{I}}, \{Y^j\}_{j \in \mathcal{J}}, \{(\theta_{i1}, \ldots, \theta_{iJ})\}_{i \in \mathcal{I}})$, where:

(i) $\mathcal{I} = \{1, \ldots, I\}$ is the set of agents;

(ii) $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ and $e^i \in \mathbb{R}^n_+$ are agent i's utility function and endowment, respectively;

(iii) $\mathcal{J} = \{1, \ldots, J\}$ is the set of firms;

(iv) $Y^j \subseteq \mathbb{R}^n$ is firm j's production technology;

(v) $\theta_{ij} \in [0, 1]$ is agent i's share of firm j, where $\sum_{i \in \mathcal{I}} \theta_{i,j} = 1$ for all $j \in \mathcal{J}$.

An allocation is a pair (x, y), where $x = (x^1, \ldots, x^I)$ is such that $x^i \in \mathbb{R}^n_+$ is agent i's consumption vector and $y = (y^1, \ldots, y^J)$ is such that $y^j \in Y^j$ is firm j's production plan. An allocation (x, y) is feasible if $\sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} e^i + \sum_{j \in \mathcal{J}} y^j$. A feasible allocation (x, y) is Pareto efficient if there exists no feasible allocation (x', y') such that $u^i(x'^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$, with strict inequality for at least one $i \in \mathcal{I}$.

A list (p^*, x^*, y^*), where $p^* \gg 0$ is a vector of prices and (x^*, y^*) is an allocation, is a Walrasian equilibrium if:

(i) for each $j \in \mathcal{J}$, $p^* \cdot y^j \leq p^* \cdot y^{*j}$ for all $y^j \in Y^j$;

(ii) for each $i \in \mathcal{I}$, x^{*i} solves

$$\begin{aligned} \max \quad & u^i(x) \\ s.t. \quad & p^* \cdot x \leq p^* \cdot e^i + \textstyle\sum_{j \in \mathcal{J}} \theta_{ij} p^* \cdot y^{*j} \ ; \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

(iii) $\sum_{i \in \mathcal{I}} x^{*i} = \sum_{i \in \mathcal{I}} e^i + \sum_{j \in \mathcal{J}} y^{*j}$.

In words, a list (p^*, x^*, y^*) is a Walrasian equilibrium if (x^*, y^*) is a feasible allocation, each firm maximizes profits given the vector of prices p^*, and each agent maximizes his utility

[3] See Chapter 16 in Mas-Colell, Whinston & Green for a proof of this result, and its appendix for a brief discussion on the Separating Hyperplane Theorem.

given the vector of prices p^* and the profits $p^* \cdot y^{*j}$ of each firm j.

As in exchange economies, it is possible to establish a counterpart of Corollary 1 on the existence of Walrasian allocations for economies with production. Likewsie, the First and Second Fundamental Welfare Theorems (with a few modifications) also hold for economies with productions[4].

5. First-Order Conditions for Pareto Optimality

Throughout this section, we consider a production economy with additional assumptions about differentiability of utility and production functions. More precisely, we assume that:

(i) Each utility function $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ is twice continuously differentiable and satisfy $\nabla u^i(x^i) \gg 0$ at all x^i. Normalize $u^i(0) = 0$.

(ii) Firm j's production set is $Y^j = \{y \in \mathbb{R}^n : F^j(y) \leq 0\}$, where $F^j : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $F^j(0) \leq 0$, and $\nabla F^j(y^j) \gg 0$ for all $y^j \in \mathbb{R}^n$.

An allocation

$$(x, y) = (x^1, \ldots, x^I, y^1, \ldots, y^J) \in \mathbb{R}^{nI}_+ \times \mathbb{R}^{nJ}$$

is Pareto optimal if it solves the following problem:

$$\begin{array}{lllr} \max & u^1(x^1_1, \ldots, x^1_n) & & \\ s.t. & u^i(x^i_1, \ldots, x^i_n) \geq \overline{u}^i, & i = 1, \ldots, I, & (1) \\ & \sum_i x^i_k \leq \sum_i e^i_k + \sum_j y^j_k, & k = 1, \ldots, n, & (2) \\ & F^j(y^j_1, \ldots, y^j_n) \leq 0, & j = 1, \ldots, J. & (3) \end{array}$$

Given the assumptions in $($i) above, we can focus attention on nonnegative utility levels $\overline{u}^i \geq 0$ for all i. Thus, finding a Pareto optimal allocation is equivalent to trying to maximize consumer 1's utility subject to (1) some minimal utility level for consumer $i \geq 2$, (2) the resource constraint in the economy, and (3) the production feasibility constraint.

Moreover, the assumptions in $($i) and $($ii) imply that all the constraints in the maximization problem above are binding at a solution. Let $(\delta_1, \ldots, \delta_I) \geq 0$, $(\mu_1, \ldots, \mu_n) \geq 0$ and $(\gamma_1, \ldots, \gamma_J) \geq 0$ be the Lagrange multipliers associated with constraints (1), (2) and (3), and

[4]See Chapter 5 in Jehle & Reny for the proof of all these results.

define $\delta_1 = 1$. We can thus write the Lagrangean

$$\max_{x,y} \quad \sum_{i=1}^{I} \delta_i u^i(x^i) + \sum_{k=1}^{n} \mu_k \left[\sum_i e_k^i + \sum_j y_k^j - \sum_i x_k^i \right] - \sum_{j=1}^{J} \gamma_j F^j(y^j)$$

and compute the first-order (Kuhn-Tucker) necessary conditions

$$x_k^i: \quad \delta_i \frac{\partial u^i}{\partial x_k^i} - \mu_k \begin{cases} \leq 0 & \text{if } x_k^i = 0 \\ = 0 & \text{if } x_k^i > 0 \end{cases},$$

$$y_k^j: \quad \mu_k - \gamma_j \frac{\partial F^j}{\partial y_k^j} = 0$$

for all i, j, k.

Now, let us assume that an interior solution ($x^i \gg 0$ for all i) is obtained. We can then use the first-order conditions above to characterize the equilibrium of this economy by means of three types of ratio conditions:

$$\frac{\partial u^i / \partial x_k^i}{\partial u^i / \partial x_{k'}^i} = \frac{\partial u^{i'} / \partial x_k^{i'}}{\partial u^{i'} / \partial x_{k'}^{i'}} \quad \text{for all } i, i', k, k',$$

$$\frac{\partial F^j / \partial y_k^j}{\partial F^j / \partial y_{k'}^j} = \frac{\partial F^{j'} / \partial y_k^{j'}}{\partial F^{j'} / \partial y_{k'}^{j'}} \quad \text{for all } j, j', k, k',$$

$$\frac{\partial u^i / \partial x_k^i}{\partial u^i / \partial x_{k'}^i} = \frac{\partial F^j / \partial y_k^j}{\partial F^j / \partial y_{k'}^j} \quad \text{for all } i, j, k, k'.$$

5.1. First-Order Conditions and the First Fundamental Welfare Theorem

If we further impose that u^i is quasiconcave for every i and that $F^j(\cdot)$ is convex for every j, then (x^*, y^*, p) is a price equilibrium with transfers with associated wealth levels $w^i = p \cdot e^i + \sum_{j \in \mathcal{J}} \theta_{ij} p \cdot y^{*j}$ if and only the first-order conditions for the decentralized budget constrained maximization problems

$$\begin{aligned} &\max_{x^i \geq 0} && u^i(x^i) \\ &s.t. && p \cdot x^i \leq w^i \end{aligned}$$

and the profit maximization problems

$$\begin{aligned} &\max_{y^j} && p \cdot y^j \\ &s.t. && F^j(y^j) \leq 0 \end{aligned}$$

are satisfied.

Let α_i and β_j denote the Lagrange multipliers for each problem. The first-order conditions evaluated at the optimum (x^*, y^*) are

$$\begin{aligned} x_k^i : &\quad \frac{\partial u^i}{\partial x_k^i} - \alpha_i p_k \begin{cases} \leq 0 & \text{if } x_k^i = 0 \\ = 0 & \text{if } x_k^i > 0 \end{cases}, \\ y_k^j : &\quad p_k - \beta_j \frac{\partial F^j}{\partial y_k^j} = 0 \end{aligned}$$

for all i, j, k. Letting $\mu_k = p_k$, $\delta_i = \frac{1}{\alpha_i}$, and $\gamma_j = \beta_j$, we obtain the first-order conditions characterizing a Pareto optimum allocation and can conclude that (x^*, y^*) is Pareto optimum if and only if it is a price equilibrium with transfers with respect to some price vector $p = (p_1, \ldots, p_n)$.

5.2. First-Order Conditions and the Second Fundamental Welfare Theorem

Let us strengthen now our assumptions about preferences by requiring that every u^i is concave. An utilitarian social planner would then choose the allocation that solves

$$\begin{array}{lll} \max_{x,y} & \sum_{i=1}^I \lambda_i u^i(x_1^i, \ldots, x_n^i) & \\ s.t. & \sum_{i=1}^I x_k^i \leq \sum_{i=1}^I e_k^i + \sum_{j=1}^J y_k^j, & k = 1, \ldots, n, \\ & F^j(y_1^j, \ldots, y_n^j) \leq 0, & j = 1, \ldots, J, \end{array}$$

where $\lambda_i > 0$ for all i. Once again, let $(\psi_1, \ldots, \psi_n)$ and $(\eta_1, \ldots, \eta_J)$ denote the Lagrange multipliers for the problem above. The necessary and sufficient first-order conditions for this problem are

$$\begin{aligned} x_k^i : &\quad \lambda_i \frac{\partial u^i}{\partial x_k^i} - \psi_k \begin{cases} \leq 0 & \text{if } x_k^i = 0 \\ = 0 & \text{if } x_k^i > 0 \end{cases}, \\ y_k^j : &\quad \psi_k - \eta_j \frac{\partial F^j}{\partial y_k^j} = 0. \end{aligned}$$

Letting $\delta_i = \frac{\lambda_i}{\lambda_1}$, $\mu_k = \frac{\psi_k}{\lambda_1}$, and $\gamma_j = \frac{\eta_j}{\lambda_1}$, we obtain the first-order conditions for a Pareto optimum allocation.