Consumer Theory

1. Primitive Notions

Consumption (Choice) Set. The set X of all alternatives (complete consumption plans) that the consumer can conceive.

Feasible Set. The subset B of X that is achievable given the constraints the consumer faces.

Consumer's Preferences. A rule specifying how the consumer ranks different alternatives.

Behavioral Assumption. The consumer seeks to identify a feasible alternative that is preferred to all other feasible alternatives.

2. Consumption Set

The set X is a non–empty closed and convex subset of $\mathbb{R}^n_+$ that contains the origin. An element of X is a consumption bundle (consumption plan). We denote a typical element of X by $x = (x_1, \ldots, x_n)$, where $x_i \geq 0$ is the amount consumed of good i.

3. Preferences

We represent consumer preferences by a binary relation $\succsim$ on X.[1] We write $x^1 \succsim x^2$ when $(x^1, x^2) \in \succsim$ and say that "x^1 is at least as good as x^2." We define strict preference and indifference as follows. If $x^1 \succsim x^2$ but $x^2 \not\succsim x^1$, we write $x^1 \succ x^2$ and say that "x^1 is (strictly) preferred to x^2." If $x^1 \succsim x^2$ and $x^2 \succsim x^1$, we write $x^1 \sim x^2$ and say that "x^1 is indifferent to x^2." Both $\succ$ and $\sim$ are binary relations on X.

Properties of $\succsim$

1. *Completeness.* The binary relation $\succsim$ is complete if for all $x^1, x^2 \in X$, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$. Notice that if $\succsim$ is complete, then $\succsim$ is *reflexive*, i.e., $x \succsim x$ for all $x \in X$.

2. *Transitivity.* The binary relation $\succsim$ is transitive if for all $x^1, x^2, x^3 \in X$, $x^1 \succsim x^2$ and

[1] Formally, a binary relation on X is a subset of $X \times X$.

$x^2 \succsim x^3$ implies that $x^1 \succsim x^3$. If $\succsim$ is transitive, then the binary relations $\succ$ and $\sim$ are also transitive (prove this).

Definition: A *preference relation* is a complete and transitive binary relation.

Properties 1 and 2 are minimal properties of preferences. Completeness implies that the consumer can compare any two alternatives. Transitivity implies that preferences rankings are internally consistent. From now on we assume that consumer preferences are represented by preference relations.

The next property of preferences is a technical condition. For each $x \in X$, let

$$\begin{aligned} \succsim(x) &= \{x' \in X : x' \succsim x\}, \\ \succ(x) &= \{x' \in X : x' \succ x\}, \\ \precsim(x) &= \{x' \in X : x \succsim x'\}, \\ \prec(x) &= \{x' \in X : x \succ x'\}. \end{aligned}$$

By construction, $\succsim(x)$ is the set of all consumption bundles that are at least as good as x, $\succ(x)$ is the set of all consumption bundles preferred to x, and so on. Note that if $\succsim$ is complete, then $\prec(x) = X \setminus \succsim(x)$ and $\succ(x) = X \setminus \precsim(x)$.

3. *Continuity.* The binary relation $\succsim$ is continuous if for each $x \in X$, the sets $\succ(x)$ and $\prec(x)$ are open subsets of X. Equivalently, $\succsim$ is continuous if for each $x \in X$, the sets $\succsim(x)$ and $\precsim(x)$ are closed subsets of X.[2]

The consumption bundle $x^* \in X$ is a satiation point of the binary relation $\succsim$ if $x^* \succsim x$ for all $x \in X$.

4. *Local Non–Satiation.* The binary relation $\succsim$ is locally non–satiated if for each $x^0 \in X$ and each $\varepsilon > 0$, there exists $x \in B_\varepsilon(x^0) \cap X$ such that $x \succ x^0$. In particular, if $\succsim$ is locally non–satiated, then $\succsim$ has no satiation point.

4'. *Strict Monotonicity.* The binary relation $\succsim$ is strictly monotonic if for each $x^0, x^1 \in X$,

[2]Recall that $S \subseteq X$ is an open subset of X for each $x \in S$ there exists $\varepsilon > 0$ such that $X \cap B_\varepsilon(x) \subseteq S$.

$x^0 \geq x^1$ implies that $x^0 \succsim x^1$ and $x^0 \gg x^1$ implies that $x^0 \succ x^1$.

4''. *Strong Monotonicity.* The binary relation $\succsim$ is strongly monotonic if for each $x^0, x^1 \in X$, $x^0 \geq x^1$ implies that $x^0 \succsim x^1$ and $x^0 > x^1$ implies that $x^0 \succ x^1$.

Note that a strongly monotonic binary relation is strictly monotonic, but the converse is not true. Likewise, a strictly monotonic binary relation is locally non–satiated, but the converse is not true.

5. *Convexity.* The binary relation is convex if $x^1 \succsim x^0$ implies that $tx^1 + (1-t)x^0 \succsim x^0$ for all $t \in [0, 1]$.

5'. *Strict Convexity.* The binary relation is strictly convex if for all $x^0, x^1 \in X$ with $x^1 \neq x^0$, $tx^0 + (1-t)x^1 \succ x^0$ for all $t \in (0, 1)$.

A strictly convex binary relation is convex, but the converse is not true. The next result provides a characterization of convex preference relations.

Lemma 1. *Suppose that $\succsim$ is a preference relation. Then $\succsim$ is convex if, and only if, $\succsim(x)$ is a convex subset of X for all $x \in X$.*

Proof: Let $\succsim$ be a preference relation. Suppose first that $\succsim$ is convex. Now let $x^0 \in X$ and consider $x^1, x^2 \in \succsim(x^0)$. Since $\succsim$ is complete, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$. Without loss, assume that $x^1 \succsim x^2$. Then $tx^1 + (1-t)x^2 \succsim x^2$ for all $t \in [0, 1]$. Since $x^2 \succsim x^0$ and $\succsim$ is transitive, we then have that $tx^1 + (1-t)x^2 \succsim x^0$ for all $t \in [0, 1]$. Hence, $\succsim(x)$ is convex.

Suppose now that $\succsim(x)$ is convex for all $x \in X$ and let $x^1, x^2 \in X$ be such that $x^1 \succsim x^2$. Then $x^1, x^2 \in \succsim(x^2)$; note that $x^2 \succsim x^2$ since $\succsim$ is reflexive (being complete). Since $\succsim(x^2)$ is convex, we then have that $tx^1 + (1-t)x^2 \in \succsim(x^2)$ for all $t \in [0, 1]$, i.e., $tx^1 + (1-t)x^2 \succsim x^2$ for all $t \in [0, 1]$. Thus, $\succsim$ is convex. □

4. Utility Functions

Definition: A utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if for all $x, x' \in X$, $u(x') \geq u(x)$ if, and only if, $x' \succsim x$.

Note that if u represents $\succsim$, then: (i) $u(x') > u(x)$ if, and only if, $x' \succ x$; (ii) $u(x') = u(x)$ if, and only if, $x' \sim x$. The next result is straightforward.

Lemma 2. *Suppose that $u : X \to \mathbb{R}$ represents $\succsim$. Then $\succsim$ is a preference relation.*

The above result says that if $\succsim$ is represented by a utility function, then $\succsim$ is complete and transitive (i.e., a preference relation). Is the converse true? In other words, can every preference relation be represented by a utility function? The next example shows that this is not the case.

Example: The lexicographic order in $\mathbb{R}^2_+$ is the binary relation $\succsim_\ell$ such that

$$(x_1^1, x_2^1) \succsim_\ell (x_1^2, x_2^2) \quad \text{if, and only if,} \quad x_1^1 > x_1^2 \text{ or } x_1^1 = x_1^2 \text{ and } x_2^1 \geq x_2^2.$$

It is straightforward to show that $\succsim_\ell$ is complete and transitive. It cannot be represented by a utility function, though. Suppose not and let $u : \mathbb{R}^2_+ \to \mathbb{R}$ represent $\succsim$. Notice that $(x, 1) \succ_\ell (x, 0)$ for all $x \in \mathbb{R}$, and so $u(x, 1) > u(x, 0)$ for all $x \in \mathbb{R}$. Let $I_x = [u(x, 0), u(x, 1)]$. Then I_x is non–empty for all $x \in \mathbb{R}$. Also notice that if $x' > x$, then $(x', 0) \succ_\ell (x, 1)$. Thus, $I_x \cap I_{x'} = \emptyset$ for all $x, x' \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each $x \in \mathbb{R}$ there exists $q_x \in \mathbb{Q} \cap I_x$. Let $f : \mathbb{R} \to \mathbb{Q}$ be such that $f(x) = q_x$. The function f is injective. Thus, $\mathbb{Q}$ and $\mathbb{R}$ have the same cardinality, a contradiction.

The lexicographic order is not continuous. For example, if $x^0 = (1, 2)$, the set $\succ_\ell (x^0)$ is not open in $\mathbb{R}^2_+$. Indeed, $x^1 = (1, 3) \in \succ_\ell (x^0)$, but $x^0 \succ_\ell (1 - \varepsilon, 3)$ for all $\varepsilon \in (0, 1)$. Thus, for all $\varepsilon > 0$, the set $B_\varepsilon(x^0) \cap \mathbb{R}^2_+$ is not contained in $\succ_\ell (x^0)$. It turns out that if $\succsim$ is a continuous preference relation, then $\succsim$ can be represented by a continuous utility function.

Theorem 1. *Suppose $\succsim$ is a continuous preference relation. Then there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.*

Proof: We establish this result when $X = \mathbb{R}^n_+$ and $\succsim$ is strictly monotonic—it is straightforward to drop the first restriction. For this, let $e = (1, \ldots, 1)$ and for each $x \in \mathbb{R}^n_+$, let $A^-(x) = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ and $A^+(x) = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$. Since $\succsim$ is strictly monotonic,

$x \succsim 0$, the zero vector. Thus, $0 \in A^-(x)$ for all $x \in \mathbb{R}^n_+$. Since $\succsim$ is continuous, we then have that $A^-(x)$ is closed and non–empty for all $x \in \mathbb{R}^n_+$. Now observe that for all $x \in \mathbb{R}^n_+$ there exists $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$. Thus, by the strict monotonicity and continuity of $\succsim$, we also have that $A^+(x)$ is non–empty and closed for each $x \in \mathbb{R}^n_+$.

Fix $x \in \mathbb{R}_+$. Since $\succsim$ is complete, we have that $\mathbb{R}_+ = A^+(x) \cup A^-(x)$. Since both $A^+(x)$ and $A^-(x)$ are closed and non–empty, we can then conclude that they have a non–empty intersection. Otherwise, $\mathbb{R}_+$ is the union of two disjoint open sets (recall that the complement of a closed set is open), which is not possible given that $\mathbb{R}_+$ is connected. Let $\alpha \in A^+(x) \cap A^-(x)$. By construction, $\alpha e \sim x$. Since $\succsim$ is strictly monotonic, the set $A^+(x) \cap A^-(x)$ must be a singleton. Let $u : X \to \mathbb{R}$ be such that $u(x)$ is the only element of $A^+(x) \cap A^-(x)$. It is easy to see that since $\succsim$ is transitive, u represents $\succsim$. For instance, suppose $u(x) \geq u(y)$. Then, by monotonicity, $u(x)e \succsim u(y)e$. Since $x \sim u(x)e$ and $y \sim u(y)e$, transitivity implies that $x \succsim y$. A similar argument shows that $x \succsim y$ implies that $u(x) \geq u(y)$.

To finish the proof, we show that u is continuous. For this, let $a < b$ be two real numbers. By construction, $u^{-1}((a,b)) = \{x \in \mathbb{R}^n_+ : u(x) > a\} \cap \{x \in \mathbb{R}^n_+ : u(x) < b\}$. Now notice, by the monotonicity and transitivity of $\succsim$, that $\{x \in \mathbb{R}^n_+ : u(x) > a\} = \{x \in \mathbb{R}^n_+ : x \succ ae\}$ and $\{x \in \mathbb{R}^n_+ : u(x) < b\} = \{x \in \mathbb{R}^n_+ : be \succ x\}$. Hence, by continuity, $u^{-1}((a,b))$ is open. Thus, u is continuous. $\square$

Let $u : X \to \mathbb{R}$ and denote the image of u by $\mathcal{U}$. The function $v : X \to \mathbb{R}$ is a monotonic transformation of u if $v(x) = \tau(u(x))$, where $\tau : \mathcal{U} \to \mathbb{R}$ is strictly increasing. Note that if u represents the preference relation $\succsim$, then v also represents $\succsim$. Indeed, $x^1 \succsim x^2$ implies that $u(x^1) \geq u(x^2)$, and so $v(x^1) = \tau(u(x^1)) \geq \tau(u(x^2)) = v(x^2)$. Now suppose that $v(x^1) \geq v(x^2)$. Since τ is strictly increasing, τ^{-1} is strictly increasing as well. Thus $\tau^{-1}(v(x_1)) = u(x_1) \geq \tau^{-1}(v(x_2)) = u(x_2)$, and so $x^1 \succsim x^2$. It turns out that if u and v are two utility functions representing $\succsim$, then v is an increasing transformation of u.

Theorem 2. *Let $\succsim$ be a preference relation on X and suppose $u : X \to \mathbb{R}$ represents $\succsim$. Then $v : X \to \mathbb{R}$ represents $\succsim$ if, and only if, v is a monotonic transformation of u.*

Proof: We know from the previous paragraph that if v is an increasing transformation of u, then v represents $\succsim$. Suppose now that v represents $\succsim$. We want to show that there exists a strictly increasing function $\tau : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U}$ is the image of u, such that $v(x) = \tau(u(x))$ for all $x \in X$. Let $f : \mathcal{U} \to X$ be such that $f(y)$ is a consumption bundle with $u(f(y)) = y$. Such a function is well–defined since $\mathcal{U}$ is the image of X under u. Now let $\tau : \mathcal{U} \to \mathbb{R}$ be such that $\tau(y) = v(f(y))$. Notice that τ is strictly increasing. Indeed, since u represents $\succsim$, $y' \geq y$ implies that $f(y') \succsim f(y)$ and $y' > y$ implies that $f(y') \succ f(y)$. Thus, since v also represents $\succsim$, $y' \geq y$ implies that $\tau(y') \geq \tau(y)$ and $y' > y$ implies that $\tau(y') > \tau(y)$. To finish, notice that $v(x) = \tau(u(x))$. Indeed, since u represents $\succsim$, $f(u(x)) \sim x$. Thus, since v also represents $\succsim$, $v(f(u(x)) = \tau(u(x)) = v(x)$. □

A function $u : X \to \mathbb{R}$ is: (i) locally non–satiated if for all $x^0 \in X$ and all $\varepsilon > 0$, there exists $x \in B_\varepsilon(x^0) \cap X$ such that $u(x) > u(x^0)$; (ii) strictly increasing if $x^1 \geq x^2$ implies that $u(x^1) \geq u(x^2)$ and $x^1 \gg x^2$ implies that $u(x^1) > u(x^2)$; (iii) strongly increasing if $x^1 \geq x^2$ implies that $u(x^1) \geq u(x^2)$ and $x^1 > x^2$ implies that $u(x^1) > u(x^2)$. Note that u is locally non–satiated if, and only if, u has no local maxima.

It is immediate to see that if u is strongly increasing, then u is strictly increasing. The converse is not true, though. Indeed, if $u : \mathbb{R}^2_+ \to \mathbb{R}$ is such that $u(x_1, x_2) = \min\{x_1, x_2\}$, then u is strictly increasing, but not strongly increasing. Likewise, if u is strictly increasing, then u is locally non–satiated, but the converse is not true. Indeed, $u : \mathbb{R}^2 \to \mathbb{R}$ such that $u(x_1, x_2) = x_1 - x_2$ is locally non–satiated, but it is not even increasing. The next result is straightforward.

Lemma 3. *Suppose $u : X \to \mathbb{R}$ represents $\succsim$. Then: (i) u is locally non–satiated if, and only if, $\succsim$ is locally non–satiated; (ii) u is strictly increasing if, and only if, $\succsim$ is strictly monotonic; (iii) u is strongly increasing if, and only if, $\succsim$ is strongly monotonic.*

The next result is also straightforward. We include its proof for completeness.

Lemma 4. *Suppose $u : X \to \mathbb{R}$ represents $\succsim$. Then: (i) u is quasi–concave if, and only if, $\succsim$ is convex; (ii) u is strictly quasi–concave if, and only if, $\succsim$ is strictly convex.*

Proof: (i) We know from Lemma 1 that if $\succsim$ is convex, then $\{x' \in X : u(x') \geq u(x)\}$ is convex for all $x \in X$. Thus, u is quasi–concave. Suppose now that u is quasi–concave and let $x^1, x^2 \in X$ be such that $x^1 \succsim x^2$. Then $u(x^1) \geq u(x^2)$ and, since u is quasi–concave, $u(tx^1 + (1-t)x^2) \geq u(x^2)$ for all $t \in [0,1]$. Thus $tx^1 + (1-t)x^2 \succsim x^2$ for all $t \in [0,1]$; i.e., $\succsim$ is convex.

(ii) Suppose that $\succsim$ is strictly convex and let $x^1 \neq x^2$. Since $\succsim$ is complete, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$. Assume, without loss, that $x^1 \succsim x^2$. Then $tx^1 + (1-t)x^2 \succ x^2$ for all $t \in (0,1)$, which implies that $u(tx^1+(1-t)x^2) > u(x^2) = \min\{u(x^1), u(x^2)\}$ for all $t \in (0,1)$. Thus, u is strictly quasi–concave. The same argument as in (i) shows that if u is strictly quasi–concave, then $\succsim$ is strictly convex. □

Suppose that $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents a convex preference relation $\succsim$ in $\mathbb{R}^2_+$ and assume that u is $\mathcal{C}^2$ and $\nabla u(x) \gg 0$ for all $x \in \mathbb{R}^2_+$.[3] In particular, u is strongly monotonic. Now let $c \in \mathcal{U}$, the image of u, and consider the indifference curve

$$u(x_1, x_2) = c. \tag{1}$$

By assumption, the set $X_1 = \{x_1 \in \mathbb{R}_+ : \exists x_2 \in \mathbb{R}_+ \text{ s.t. } u(x_1, x_2) = c\}$ is non–empty. Moreover, since $\partial u(x)/\partial x_2 > 0$, for each $x_1 \in X_1$, there exists a unique $x_2 \in \mathbb{R}_+$ such that $u(x_1, x_2) = c$. In other words, (1) defines a function $f : X_1 \to \mathbb{R}_+$ such that $u(x_1, f(x_1)) = c$ for all $x_1 \in X_1$. The graph of f is the indifference curve of u with (utility) level c.

By the Implicit Function Theorem, we have that

$$f'(x_1) = -\frac{\dfrac{\partial u}{\partial x_1}(x_1, f(x_1))}{\dfrac{\partial u}{\partial x_2}(x_1, f(x_1))} = -\mathrm{MRS}_{1,2}(x_1, f(x_1)) \tag{2}$$

for all $x_1 \in X_1$; $\mathrm{MRS}_{1,2}(x_1, x_2)$ is the marginal rate of substitution between goods 1 and 2 at the bundle (x_1, x_2). Now observe, by (2) and the Chain rule, that

$$\begin{aligned} f''(x_1) \;\; \propto \;\; & -\frac{\partial^2 u}{\partial x_1^2}(x_1, f(x_1))\frac{\partial u}{\partial x_2}(x_1, f(x_1)) - \frac{\partial^2 u}{\partial x_2 \partial x_1}(x_1, f(x_1)) f'(x_1) \frac{\partial u}{\partial x_2}(x_1, f(x_1)) \\ & + \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1, f(x_1))\frac{\partial u}{\partial x_1}(x_1, f(x_1)) + \frac{\partial^2 u}{\partial x_2^2}(x_1, f(x_1)) f'(x_1) \frac{\partial u}{\partial x_1}(x_1, f(x_1)). \end{aligned}$$

[3]Implicit in this argument is the assumption that u is defined in a neighborhood of $\mathbb{R}^2_+$

Using (2) again (and omitting the algebra), we have that

$$f'(x_1) \propto -h' Hu(x_1, f(x_1))h,$$

where $h = (-\partial u(x_1, f(x_1))/\partial x_2, \partial u(x_1, f(x_1))/\partial x_1)$. Since $\nabla u(x_1, f(x_1)) \cdot h = 0$ and u is quasi–concave, we then have that $h' Hu(x_1, f(x_1))h \leq 0$. Thus, $f'(x_1) \geq 0$, which implies that the marginal rate of substitution between goods 1 and 2 is decreasing in x_1. This argument can be extended to utility functions with more than two arguments.

5. The Consumer's Problem

Let $p = (p_1, \ldots, p_n) \gg 0$ be the vector of prices and $w > 0$ be the consumer's wealth. The problem of the consumer is to find $x^* \in \mathcal{B}(p, w) = \{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq w\}$ such that $x^* \succsim x$ for all $x \in \mathcal{B}(p, w)$. The set $\mathcal{B}(p, w)$ is the budget set given p and w. If $\succsim$ is a continuous preference relation on $\mathbb{R}^n_+$, there exists a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that represents $\succsim$. In this case, the consumer's problem is

$$\begin{array}{ll} \max & u(x) \\ \text{s.t.} & w - \langle p, x \rangle \geq 0 \quad (CP) \\ & x \geq 0 \end{array}$$

We assume that $\succsim$ is continuous, and so is represented by a continuous utility function.

Existence.

Notice that for all $p \gg 0$ and $w > 0$, the set $\mathcal{B}(p, w)$ is closed and bounded. Indeed, let $\{x^k\}$ be a sequence in $\mathcal{B}(p, w)$ that converges to x. First, notice that since $x^k \geq 0$ for all $k \geq 1$, $x \geq 0$ as well. Now notice that $g_0(x) = w - \langle p, x \rangle$ is continuous in x, and so $g_0(x^k)$ converges to $g_0(x)$. Since $g_0(x^k) \geq 0$ for all k, $g_0(x) \geq 0$ as well. Thus, $x \in \mathcal{B}(p, w)$ so that $\mathcal{B}(p, w)$ is closed. To finish, notice that if $x \in \mathcal{B}(p, w)$, then $x_i \geq 0$ and $x_i \leq w/p_i$ for each $i \in \{1, \ldots, n\}$. Thus, $\mathcal{B}(p, w)$ is bounded. Since u is continuous, we can then conclude that (CP) has a solution for all $p \gg 0$ and $w > 0$.

Uniqueness.

The set $\mathcal{B}(p,w)$ is convex for all $p \gg 0$ and $w > 0$. So, (CP) has a unique solution if $\succsim$ is strictly convex, since then u is strictly quasi-concave. From now on we assume that $\succsim$ is strictly convex. Then it is meaningful to talk about the consumer's (Marshalian) demand function $x(p,w)$: it is the solution to (CP) when the vector of prices is $p \gg 0$ and the consumer's wealth is $w > 0$.

Let $v(p,w) = u(x(p,w))$ be the indirect utility function; notice that v is well-defined even if the solution to the consumer's problem is not unique. We have the following result.

Theorem 3. *The functions $x(p,w)$ and $v(p,w)$ have the following properties:*

(i) $x(p,w)$ and $v(p,w)$ are continuous;

(ii) $x(p,w)$ and $v(p,w)$ are homogeneous of degree zero;

(iii) $v(p,w)$ is increasing in w and decreasing in p. If $\succsim$ is locally non-satiated, v is strictly increasing in w and strictly decreasing in p.

Proof: (i) This follows immediately from the Maximum Theorem together with the fact that $x(p,w)$ is a single–valued upper hemicontinuous correspondence.

(ii) This follows immediately from the fact that $\mathcal{B}(p,w) = \mathcal{B}(\alpha p, \alpha w)$ for all $\alpha > 0$.

(iii) Notice that if $w' > w$, then $\mathcal{B}(p,w) \subseteq \mathcal{B}(p,w')$. In particular, if x^* is the optimal consumption bundle when the consumer's wealth is w, then x^* is feasible when the consumer's wealth is w'. Hence, $v(p,w') \geq u(x^*) = v(p,w)$. Now suppose that $\succsim$ is locally non–satiated and let $\varepsilon > 0$ be such that $\mathcal{B}_\varepsilon(x^*) \subseteq \mathcal{B}(p,w')$. Such an ε exists since $w' > w$, and so x^* costs less than w'. Since $\succsim$ is locally non–satiated, there exists $x' \in \mathcal{B}_\varepsilon(x^*)$ such that $u(x') > u(x^*)$. Thus, $v(p,w') \geq u(w') > u(x^*) = v(p,w)$. A similar argument shows that $v(p,w)$ is decreasing in p and strictly decreasing in p if $\succsim$ is locally non–satiated. $\square$

6. The Expenditure Minimization Problem

Let $p = (p_1, \ldots, p_n) \gg 0$ and $\mathcal{U} = \{u(x) : x \in \mathbb{R}^n_+\}$ denote the set of attainable utility levels. For each $\overline{u} \in \mathcal{U}$, the expenditure minimization problem is

$$\begin{aligned} \min \quad & \langle p, x\rangle \\ \text{s.t.} \quad & u(x) \geq \overline{u} \quad (EMP) \\ & x \geq 0 \end{aligned}$$

Existence.

Notice that for all $p \gg 0$ and $x \in \mathbb{R}^n_+$, $\langle p, x\rangle \geq 0$, and thus the set of numbers $\{e : x \in \mathbb{R}^n_+, u(x) \geq \overline{u}, e = \langle p, x\rangle\}$ is bounded below by zero. In a similar fashion to the consumer's problem, this set is also closed since $p \gg 0$ and u is assumed to be continuous. Hence, it contains a smallest number, namely $e(p, \overline{u})$.

Uniqueness.

Since $\succsim$ is assumed to be strictly convex, the set $\{x \in \mathbb{R}^n_+ : u(x) \geq 0\}$ is convex. Moreover, because $u(x)$ is continuous and strictly quasi–concave, *(EMP)* has a unique solution.

The solution $x^h(p, \overline{u})$ is the Hicksian (or compensated) demand correspondence, and $e(p, \overline{u}) = \langle p, x^h(p, \overline{u})\rangle$ is the expenditure function.

Theorem 4. *Suppose u is continuous and strictly quasi–concave. The following holds:*

(i) if u is also locally non–satiated, then $x^h(p, \overline{u})$ and $e(p, u)$ are continuous;

(ii) $x^h(p, \overline{u})$ is homogenous of degree zero in p, and $e(p, \overline{u})$ is homogeneous of degree 1 in p;

(iii) $e(p, \overline{u})$ is strictly increasing in $\overline{u}$ and increasing in p;

(iv) (Shephard's lemma): *if $u(x)$ is strictly quasi–concave, then $e(p, \overline{u})$ is differentiable in p at (p^0, u^0) with $p^0 \gg 0$ and, for $i \in \{1, \ldots, n\}$,*

$$\frac{\partial e(p^0, u^0)}{\partial p_i} = x_i^h(p^0, u^0).$$

Proof: (i) Follows from the Maximum Theorem.

(ii) Homogeneity follows from the fact that the optimal vector when minimizing $\langle p, x\rangle$ subject to $u(x) \geq \overline{u}$ is the same as that for minimizing $\langle \alpha p, x\rangle$ subject to the same constraint, for any scalar $\alpha > 0$.

(iii) Suppose, by contradiction, that $e(p,\overline{u})$ is not strictly increasing in $\overline{u}$, and let x' and x'' denote optimal consumption bundles for utility levels u' and u'', respectively, where $u'' > u'$ and $\langle p, x'\rangle \geq \langle p, x''\rangle > 0$. Consider a bundle $\widetilde{x} = \alpha x''$, where $\alpha \in (0,1)$. By continuity of $u(\cdot)$, there exists an α close enough to 1 such that $u(\widetilde{x}) > u'$ and $\langle p, x'\rangle > \langle p, \widetilde{x}\rangle$. But this contradicts x' being an optimal consumption bundle in the (EMP) with required utility level u'.

To show that $e(p,\overline{u})$ is increasing in p_l, consider price vectors p'' and p' such that $p''_l \geq p'_l$ and $p''_k = p'_k$ for all $k \neq l$. Let x'' be the solution to the (EMP) for prices p''. Then $e(p'',u) = \langle p'', x''\rangle \geq \langle p', x''\rangle \geq e(p',\overline{u})$, where the latter inequality follows from the definition of $e(p',\overline{u})$.

(iv) This is a direct consequence of the Envelope Theorem applied to the (EMP). In particular, because prices are parameters in the (EMP) that enter only on the objective function $\langle p, x\rangle$, the change in the value function of the (EMP) with respect to a change in prices is just the partial derivative with respect to p of the objective function evaluated at the optimal vector $x^h(p,\overline{u})$. □

7. Duality

Suppose u is continuous, so that: *(i)* the indirect utility function $v(p,w)$ is well–defined for all $p \gg 0$ and $w \geq 0$; *(ii)* the expenditure function $e(p,u)$ is well–defined for all $p \gg 0$ and $\overline{u} \in \mathcal{U}$. First note that if $x(p,w)$ is a solution to the consumer's problem given prices $p \gg 0$ and wealth $w \geq 0$, then $\langle p, x(p,w)\rangle \leq w$ and $u(x(p,w)) = v(p,w) \geq \overline{u}$. Therefore,

$$e(p, v(p,w)) \leq \langle p, x(p,w)\rangle \leq w$$

for all $p \gg 0$ and $w \geq 0$. Now observe that if $x^h(p,\overline{u})$ is a solution to the expenditure minimization problem given prices $p \gg 0$ and utility level $\overline{u} \in \mathcal{U}$, then $u(x^h(p,\overline{u})) \geq \overline{u}$ and $\langle p, x^h(p,\overline{u})\rangle \leq e(p,\overline{u})$. Therefore,

$$v(p, e(p,\overline{u})) \geq u(x^h(p,\overline{u})) \geq \overline{u}$$

for all $p \gg 0$ and $\overline{u} \in \mathcal{U}$. The following results show that a stronger statement is possible when u is also strictly increasing.

Theorem 5. *Let $v(p, w)$ and $e(p, u)$ be the indirect utility function and the expenditure function, and suppose that u is continuous and strictly increasing. Then for all $p \gg 0$, $w \geq 0$ and $\overline{u} \in \mathcal{U}$:*

(i) $e(p, v(p, w)) = w$.

(ii) $v(p, e(p, \overline{u})) = \overline{u}$.

Proof: Recall that u strictly increasing implies that $\mathcal{U} = [u(0), \overline{U})$, where $\overline{U}$ is the supremum of u in $\mathbb{R}^n_+$.

(i) Let $(p, w) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$ and suppose, by contradiction, that $e(p, v(p, w)) < w$; we know that $e(p, v(p, w)) \leq w$. Now let $\overline{u} = v(p, w)$. Since $\overline{u} < \overline{U}$ and $e(p, \overline{u})$ is continuous (given that u is locally non–satiated), there exists $\varepsilon > 0$ with $\overline{u} + \varepsilon < \overline{U}$ such that $e(p, \overline{u} + \varepsilon) < w$. Note that $v(p, e(p, \overline{u} + \varepsilon)) \geq \overline{u} + \varepsilon$. However, since $v(p, w)$ is strictly increasing in w (using once again the fact that u is locally non–satiated), we can then conclude that $\overline{u} = v(p, w) \geq v(p, e(p, \overline{u} + \varepsilon)) \geq \overline{u} + \varepsilon$, a contradiction.

(ii) Let $(p, \overline{u}) \in \mathbb{R}^n_{++} \times \mathcal{U}$ and suppose, by contradiction, that $v(p, e(p, \overline{u})) > \overline{u}$. Note that if $\overline{u} = u(0)$, then $e(p, \overline{u}) = 0$, and so $v(p, e(p, \overline{u})) = v(p, 0) = u(0)$. Hence, it must be that $\overline{u} > u(0)$. Now let $w = e(p, \overline{u})$. Note that $w > 0$. Since $v(p, w)$ is continuous, there exists $\varepsilon > 0$ such that $w - \varepsilon > 0$ and $v(p, w - \varepsilon) > \overline{u}$. However, since $e(p, \overline{u})$ is increasing in $\overline{u}$, we then have that $w = e(p, \overline{u}) \leq e(p, v(p, w - \varepsilon)) = w - \varepsilon$, a contradiction. □

It is not difficult to see that the above result is true if u is locally non–satiated. We can now prove a result that will play a key role in the derivation of the Slutsky equation.

Theorem 6. *Suppose that u is continuous, strictly quasi–concave, and locally non–satiated. Then, for all $p \gg 0$, $w \geq 0$ and $\overline{u} \in \mathcal{U}$:*

(i) $x_i(p, w) = x_i^h(p, v(p, w))$.

(ii) $x_i^h(p, \overline{u}) = x_i(p, e(p, \overline{u}))$.

Proof: (i) Let $x^0 = x(p^0, w^0)$, where $p^0 \gg 0$ and $w^0 \geq 0$. Moreover, let $\overline{u}^0 = u(x^0)$, so that $v(p^0, w^0) = \overline{u}^0$. Since u is locally non–satiated, $\langle p^0, x^0 \rangle = w^0$. Now observe that $e(p^0, \overline{u}^0) =$

w^0 by the previous theorem. Since $u(x^0) \geq \overline{u}^0$ and $\langle p^0, x^0 \rangle = e(p^0, \overline{u}^0)$, we can then conclude that $x^0 = x^h(p^0, \overline{u}^0)$; given that u is strictly quasi–concave and continuous, the solution to the expenditure minimization problem is unique. Thus, $x(p^0, w^0) = x^h(p^0, v(p^0, w^0))$.

(ii) Prove this. □

Fix the utility level $u^* = v(p, w)$ a consumer obtains when facing prices $p \gg 0$ and wealth $w > 0$. From the previous Theorem, we know that

$$x_i^h(p, u^*) = x_i(p, e(p, u^*)).$$

Differentiating both sides with respect to p_j yields

$$\frac{\partial x_i^h(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, e(p, u^*))}{\partial p_j} + \frac{\partial x_i(p, e(p, u^*))}{\partial w} \frac{\partial e(p, u^*)}{\partial p_j}. \tag{3}$$

Now, recall that by assumption $e(p, u^*) = e(p, v(p, w)) = w$, and therefore, $\frac{\partial e(p,u^*)}{\partial p_j} = x_j^h(p, u^*) = x_j^h(p, v(p, w)) = x_j(p, w)$. Thus, (3) can be rewritten as

$$\frac{\partial x_i^h(p, u^*)}{\partial p_j} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w),$$

which we can rearrange into the *Slutsky equation*

$$\underbrace{\frac{\partial x_i(p, w)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial x_i^h(p, u^*)}{\partial p_j}}_{SE} \underbrace{- x_j(p, w) \frac{\partial x_i(p, w)}{\partial w}}_{IE}.$$

The Slutsky equation decomposes the total effect *(TE)* a change in price of good j has over the Marshallian demand of good i in two parts, the substitution (*SE*) and income (*IE*) effects.

Another interesting result can be obtained by the second equality in Theorem 5. Let $u^* = v(p^*, w^*)$. Because $v(p, e(p, u^*)) = u^*$ for all p, differentiating with respect to p_j and evaluating at $p = p^*$ yields

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} \frac{\partial e(p^*, u^*)}{\partial p_j} = 0.$$

But Shephard's lemma implies that $\frac{\partial e(p^*,u^*)}{\partial p_j} = x_j^h(p^*, u^*)$, which we substitute to obtain

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} x_j^h(p^*, u^*) = 0.$$

Last, since $w^* = e(p^*, u^*)$ by the first equality of Theorem 5, the second equality of Theorem 6, and the definition of u^*, we can write

$$\frac{\partial v(p^*, w^*)}{\partial p_j} + \frac{\partial v(p^*, w^*)}{\partial w} x_j(p^*, u^*) = 0,$$

which we rearrange to obtain Roy's identity

$$x_j(p^*, w^*) = -\frac{\frac{\partial v(p^*, w^*)}{\partial p_j}}{\frac{\partial v(p^*, w^*)}{\partial w}}.$$