Theory of the Firm

1. Production Technology

Suppose there are m goods in the economy. A production plan is a vector $y = (y_1, \ldots, y_m)$ in $\mathbb{R}^m$, where $y_i > 0$ means good i is an output and y_i is the amount produced and $y_i < 0$ means good i is an input and $-y_i$ is the amount used as an input. The production set is the set Y of all feasible production plans. It is customary to describe the production set by means of a transformation function $F : \mathbb{R}^m \to \mathbb{R}$ as follows: $Y = \{y \in \mathbb{R}^m : F(y) \leq 0\}$. The set $\{y \in \mathbb{R}^m : F(y) = 0\}$ is the transformation frontier of Y.

Example: $F(z, q_1, \ldots, q_n) = z - f(q_1, \ldots, q_n)$, where $f : \mathbb{R}^n_+ \to \mathbb{R} \bigcup \{-\infty\}$ is such that if $q = (q_1, \ldots, q_n) \notin \mathbb{R}^n_+$, then $f(q) = -\infty$. Thus, goods 2 to n can only be used as inputs to production. The restriction of f to $\mathbb{R}^n_+$ is called the production function of this technology.

In what follows we give a list of common assumptions about the production set Y.

(i) Y is non–empty. This property is self–evident.

(ii) Y closed. This is a technical assumption.

(iii) Possibility of inaction: $0 \in Y$. The firm can shut down completely. This is possible if there are no sunk costs, i.e., there are no commitments to the use of some inputs to production.

(iv) No Free Lunch: $Y \bigcap \mathbb{R}^m_+ \subseteq \{0\}$. This implies that there is no $y > 0$ in Y. In other words, there is no feasible production plan where some goods are produced with no inputs.

(v) Free Disposal: $Y - \mathbb{R}^m_+ \subseteq Y$. Notice that free disposal implies that if $y \in Y$, then $y' \in Y$ for all $y' \leq y$. A production plan $y' \leq y$ means producing at most the same amounts of outputs as in y with at least as much of the inputs.

(vi) Irreversibility: $Y \bigcap -Y \subseteq \{0\}$. Irreversibility implies that if $y \in Y$ and $y \neq 0$, then $-y \notin Y$. In other words, switching the roles of inputs and outputs in a feasible production

plan is never feasible.

(vii) Non–Increasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \in [0,1]$. Every production plan can be scaled down. In particular, inaction is possible.

(viii) Non–Decreasing Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 1$. Every production plan can be scaled up. A weaker version of non–decreasing returns to scale is to assume that $Y+Y \subseteq Y$, which is called *free entry*. Free entry implies that if y is feasible, then replicating Y any number of times is also feasible, i.e., if y is feasible, then ny is feasible for all $n \in \mathbb{Z}_+$.

(ix) Constant Returns to Scale: $\alpha Y \subseteq Y$ for all $\alpha \geq 0$.

(x) Convexity: Y is convex. Notice that if inaction is possible, then convexity implies non–increasing returns to scale. Indeed, if Y is convex, then $\alpha y = \alpha y + (1-\alpha)0 \in Y$ for all $\alpha \in [0,1]$.

2. Profit Maximization

Suppose $Y = \{y \in \mathbb{R}^m : F(y) \leq 0\}$, where $F : \mathbb{R}^m \to \mathbb{R}$ is such that:

(i) F is continuous;

(ii) F is quasi-convex;

(iii) $F(0) = 0$;

(iv) There exists $\overline{y} \in \mathbb{R}^m$ such that $F(y) \leq 0$ implies that $y \leq \overline{y}$.

Then, Y is a closed and convex set that contains the origin. Moreover, Y is bounded above by $\overline{y}$. Notice that *(ii)* implies that $\overline{y}$ must be in $\mathbb{R}^m_+$. Also notice that *(iv)* rules out constant returns to scale.

Let $p \gg 0$ be the vector of prices. The problem of the firm is then

$$\begin{aligned} &\max\nolimits_y \langle p, y\rangle \\ &s.t. - F(y) \geq 0 \end{aligned} \tag{PM}$$

Since $F(0) = 0$ and $y \leq \overline{y}$ for all $F(y) \leq 0$, adding the constraints $\langle p, y\rangle \geq 0$ and $y \leq \overline{y}$

to *(PM)* does not alter the problem. The constraint set in the modified problem is closed and bounded, though. Thus, *(PM)* has a solution.

Lemma 1. *The profit maximization problem has a solution.*

Let

$$\pi(p) = \max\{\langle p, y\rangle : F(y) \leq 0\}$$

be the profit function and

$$y(p) = \{y \in \mathbb{R}^m : F(y) \leq 0 \text{ and } \langle p, y\rangle = \pi(p)\}$$

be the supply correspondence. The constraint correspondence is $\Gamma : \mathbb{R}^m_{++} \rightrightarrows \mathbb{R}^m$ given by $\Gamma(p) = \{y \in \mathbb{R}^m : F(y) \leq 0\}$. We know that in general the set $\{y \in \mathbb{R}^m : F(y) \leq 0\}$ is not compact, and thus Γ will fail to be upper hemicontinuous. However, if $\Gamma_* : \mathbb{R}^m_{++} \rightrightarrows \mathbb{R}^m$ is such that $\Gamma_*(p) = \{y \in \mathbb{R}^m : F(y) \leq 0, \langle p, y\rangle \geq 0, y \leq \overline{y}\}$, then

$$\pi(p) = \max\{\langle p, y\rangle : y \in \Gamma_*(p)\}$$

and

$$y(p) = \{y \in \Gamma_*(p) : \langle p, y\rangle = \pi(p)\}.$$

One can show that Γ_* is a continuous correspondence. Thus, by the Maximum Theorem, $\pi(p)$ is continuous and $y(p)$ is upper hemicontinuous. Now, assume that

(ii') F is strictly quasi-convex.

In this case it is easy to see that the solution to *(PM)* is unique, so that $y(p)$ is a function. We thus have the following result.

Lemma 2. *The profit maximization problem has a unique solution. Both the profit function and the supply function are continuous.*

We want to be able to use Kuhn–Tucker to analyze the profit maximization problem.

For this, suppose now that:

(i') F is differentiable;

(ii") F is differentiably strictly quasi-convex;

(v) $\nabla F(y) \gg 0$ for all $y \in \mathbb{R}^m$.

A consequence of *(v)* and the quasi–convexity of F is that Y has no free lunch.

Lemma 3. *Y has no free lunch.*

Proof: Since F is quasi–convex, $F(y') \geq F(y)$ implies that $\langle \nabla F(y'), y - y' \rangle \leq 0$. Thus, $F(y) \leq 0 = F(0)$ implies that $\langle \nabla F(0), y \rangle \leq 0$, from which we can conclude that $y \not> 0$. In other words, $Y \bigcap \mathbb{R}^m_+ = \{0\}$. $\square$

Notice that $-F$ is pseudo–concave. Since $\nabla F(0) \gg 0$, $-F(\varepsilon, \dots, \varepsilon) > 0$ if ε is small enough. In other words, there exists $\widehat{y} \in \mathbb{R}^m$ such that $-F(\widehat{y}) > 0$. Thus, the Kuhn–Tucker conditions are necessary. Moreover, since $\langle p, y \rangle$ is pseudo–concave, the Kuhn–Tucker conditions are sufficient as well. The Kuhn–Tucker *(KT)* conditions for *(PM)* are

$$\begin{aligned} p - \lambda \nabla F(y) &= 0 \\ -\lambda F(y) &= 0, \end{aligned}$$

with $\lambda \geq 0$ and $F(y) \leq 0$.

Let y^* denote the unique solution to *(KT)*. If $F(y^*) < 0$, then $\lambda = 0$, in which case we have $p = 0$, a contradiction. Thus, $F(y^*) = 0$ and $\lambda > 0$. We then have the following result.

Theorem 1. *Suppose* (i'), (ii"), (iii), (iv), *and* (v) *are satisfied. For each $p \gg 0$, the solution to the profit maximization problem is unique and completely characterized by the Kuhn–Tucker conditions*

$$\begin{aligned} &p = \lambda \nabla F(y) \textit{with } \lambda > 0, \\ &F(y) = 0. \end{aligned}$$

Theorem 2. *Suppose that* (iii)*,* (iv)*, and* (v) *are satisfied. Moreover, suppose that F is C^2 and such that $h'HF(y)h > 0$ for all $y \in \mathbb{R}^m$ and $h \in \mathbb{R}^m \setminus \{0\}$ such that $\langle \nabla F(y), h \rangle = 0$. Then the supply function is differentiable.*

Proof: Let $G : \mathbb{R}^M \times \mathbb{R}_{++} \times \mathbb{R}^m_{++} \to \mathbb{R}^m \times \mathbb{R}$ be such that

$$G(y, \lambda, p) - (p - \lambda \nabla F(y), -F(y)).$$

Notice that *(ii')* is satisfied, and so Theorem 1 implies that $G(y, \lambda, p) = 0$ implicitly defines y (and λ) as functions of p. The desired result holds by the Implicit Function Theorem if the Jacobian matrix $DG_{(y,\lambda)}(y, \lambda, p)$ has full rank. For this, notice that

$$DG_{(y,\lambda)}(y, \lambda, p) = \begin{bmatrix} -\lambda HF(y) & -\nabla F(y)' \\ -\nabla F(y) & 0 \end{bmatrix},$$

where here we treat $\nabla F(y)$ as a row vector for purposed of matrix multiplication, and consider the following homogeneous linear system

$$DG_{(y,\lambda)}(y, \lambda, p) \begin{bmatrix} \Delta y \\ \Delta \lambda \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -\lambda HF(y)\Delta y + \nabla F(y)' \Delta \lambda = 0 \\ \langle \nabla F(y), \Delta y \rangle = 0 \end{cases}$$

The Jacobian matrix $DG_{(y,\lambda)}$ has full rank if $\Delta y = \Delta \lambda = 0$ is the only solution to the above system. Suppose, by contradiction, that $\Delta y \neq 0$. Then,

$$\lambda \Delta y' HF(y) \Delta y + \langle \nabla F(y), \Delta y \rangle \Delta y = \lambda \Delta y' HF(y) \Delta y = 0,$$

which implies, since $\lambda > 0$, that $0 < \Delta y' HF(y) \delta y = 0$, a contradiction. Thus, $\Delta y = 0$, from which we can conclude that $\Delta \lambda = 0$. $\square$