Problem Set 3

Problem 1

Given a production set $Y \subset \mathbb{R}^m$, let $\pi : \mathbb{R}^m_{++} \to \mathbb{R}$ be the profit function given by $\pi(p) = \sup\{\langle p, y\rangle : y \in Y\}$. Show that:

(a) π is homogeneous of degree 1.

(b) If Y is convex, then π is convex.

(c) If Y has non-decreasing returns to scale, then, $\forall p \gg 0$, either $\pi(p) \leq 0$ or $\pi(p) = \infty$.[1]

Now, let $y : \mathbb{R}^m_{++} \rightrightarrows Y$ be the supply correspondence such that $y(p) = \{y \in Y : \langle p, y\rangle = \pi(p)\}$. Show that:

(d) $y(\alpha p) = y(p)$ for all $p \gg 0$ and all $\alpha > 0$.

(e) If Y is convex, $y(p)$ is convex for every $p \gg 0$.

(f) y satisfies the Law of Supply, i.e.

$$\langle p - p', y - y'\rangle \geq 0$$

for all $p, p' \in \mathbb{R}^m_{++}$ and all $y, y' \in Y$ such that $y \in y(p)$ and $y' \in y(p')$.

Problem 2

Consider a firm with production technology with a single output and $n \geq 2$ inputs, and let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be the production function associated with this technology. Thus, the production plan $(z, q_1, \ldots, q_n) \in \mathbb{R} \times \mathbb{R}^n_-$ is feasible if, and only if, $z \leq f(x_1, \ldots, x_n)$, where $x_i = -q_i$ is the consumed quantity of input i. Let $w \gg 0$ be the input price vector, and consider the cost minimization problem

$$\begin{aligned} \min_x \quad & \langle w, x\rangle \\ s.t. \quad & f(x) \geq q \quad (CM) \end{aligned}$$

Suppose that f is continuous, $f(0) = 0$, and that for every $q > 0$ there exists $x \in \mathbb{R}^n_+$ with $f(x) \geq q$. Thus, (CM) has a solution for every $(w, q) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, and the function

[1] Be mindful here because π take values in $\mathbb{R} \bigcup \{\infty\}$. A function $f : \mathbb{R}^m_{++} \to \mathbb{R} \bigcup \{\infty\}$ is homogenous of degree k if $f(\alpha p) = \alpha^k f(p)$ for all $\alpha > 0$ and all $p \gg 0$.

$c : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ given by $c(w, q) = \min\{\langle w, x\rangle : f(x) \geq q\}$ is well-defined. We say c is the firm's cost function. Show that:

(a) c is homogenous of degree 1 in w and non-decreasing in q.

(b) c is concave in w.

(c) If f is homogenous of degree 1, then c is homogenous of degree 1 in q.

Problem 3 (Jehle & Reny, problem 3.53)

A utility produces electricity to meet the demands of a city. The price it can charge for electricity is fixed and it must meet all demand at that price. It turns out that the amount of electricity demanded is always the same over every 24-hour period, but demand differs from day (6:00h to 18:00h) and night (18:00h to 6:00h). During the day, 4 units are demanded, whereas during the night only 3 units are demanded. Total output for each 24-hour period is thus always equal to 7 units. The utility produces electricity according to the production function

$$y_i = (KF_i)^{1/2}, \quad i\text{=day, night},$$

where K is the size of the generating plant, and F_i is tons of fuel. The firm must build a single plant; it cannot change plant size from day to night. If a unit of plant costs w_k per 24-hour period and a ton of fuel costs w_f, what size of plant will the utility build?

Problem 4

Suppose that one output can be produced from two inputs according to many different technologies, where every technology can produce up to one unit of output with fixed and proportional inputs z_1 and z_2, but no more than that. In other words, the production technologies can be described by the probability density function $g(z_1, z_2)$.

(a) If the density is a uniform distribution on $[0, 10] \times [0, 10]$, solve the firm's profit maximization problem when input prices are $w = (w_1, w_2)$ and the output price is normalized to 1 and compute the firm's aggregate profit.

(b) Calculate the firm's aggregate production function. Are there any restrictions in the input prices?

Problem 5

There is a firm with a single-output technology characterized by the production function f, and let $c(\mathbf{w}, y)$ be the associated cost function. For price vector $(p, \mathbf{w}) \gg 0$, where p is the price for output y and $\mathbf{w}$ is the price vector for inputs $\mathbf{z} \in \mathbf{R}_+^m$, consider the problems

$$\max_{y \geq 0} \quad py - c(\mathbf{w}, y) \tag{1}$$

with solution $y^* \geq 0$ and

$$\max_{\mathbf{z} \in \mathbf{R}_+^m} \quad pf(\mathbf{z}) - \langle \mathbf{w}, \mathbf{z} \rangle \tag{2}$$

with solution $\mathbf{z}^* \geq 0$.

(a) Show that $\widehat{y} = f(\mathbf{z}^*)$ solves (1).

(b) Show that if $c(\mathbf{w}, y^*) = \langle \mathbf{w}, \widehat{\mathbf{z}} \rangle$ and $y^* = f(\widehat{\mathbf{z}})$, then $\widehat{\mathbf{z}}$ solves (2).

(c) Use parts (a) and (b) to show that $py^* - c(\mathbf{w}, y^*) = pf(\mathbf{z}^*) - \langle \mathbf{w}, \mathbf{z}^* \rangle$.

Problem 6

Consider a production possibility set $Y = \{y \in \mathbb{R}^m : F(y) \leq 0\}$ where $F : \mathbb{R}^m \to \mathbb{R}$ is such that

(i) F is continuous;

(ii) F is quasi-convex;

(iii) $F(0) = 0$.

Show that a firm producing a single output using such a technology can be described by a concave production function $f(z)$, where $y = (q, -z_1, \ldots, -z_{m-1})$.

Exercises

Solve exercises 5.C.7 and 5.E.3 from Mas-Colell, Whinston and Green.