Problem Set - Choice under Uncertainty

Problem 1

Let $\succsim$ be a binary relation in $\mathcal{G}$. We say that $\succsim$ satisfies the Independence Axiom if

$$(p_1 \circ a_1, \dots, p_n \circ a_n) \sim (q_1 \circ a_1, \dots, q_n \circ an\_n)$$

implies that, for all $\alpha \in [0,1]$ and any gamble $(r_1 \circ a_1, \dots, r_n \circ a_n)$,

$$\begin{aligned}((\alpha p_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha p_n + (1-\alpha) r_n) \circ a_n)\\ \sim ((\alpha q_1 + (1-\alpha) r_1) \circ a_1, \dots, (\alpha q_n + (1-\alpha) r_n) \circ a_n).\end{aligned}$$

Show that if $\succsim$ satisfies the Substitution and Reduction to Simple Gambles Axioms, then $\succsim$ satisfies the Independence Axiom.

Problem 2

There are three possible outcomes: €5, €10 and €15. Consider the following gambles: $g_1 = 1 \circ 10$, $g_2 = (1/2 \circ 5, 1/2 \circ 15)$, and $g_3 = (1/3 \circ 5, 1/3 \circ 10, 1/3 \circ 15)$. Is it possible that an agent who prefers g_3 to g_2 and g_2 to g_1 be an expected utility maximizer?

Problem 3

Consider an agent endowed with an utility function u over $\mathbb{R}^n_+$.

(a) Show that the concavity of u can be interpreted as the agent exhibiting risk-aversion with respect to gambles whose prizes are consumption bundles in $\mathbb{R}^n_+$.

(b) Fix a price vector and let $v : \mathbb{R}_+ \to \mathbb{R}$ the indirect utility function of an agent as a function of his/her wealth. Show that if u is concave, then v is also concave. Interpret.

Problem 4

Suppose there are two assets. The first one is a risk-free asset that pays €1. The second is a risky asset that pays €a with probability π and €b with probability $1 - \pi$. The price of each asset is €1. Consider a risk-averse agent whose initial wealth is €1.

(a) Derive a necessary condition (on a and b) for the risk-free asset's demand to be positive.

(b) Derive a necessary condition (on a, b, and π) for the risky asset's demand to be positive.

Suppose that the conditions derived in (a) and (b) are satisfied.

(c) Determine the first-order conditions for the agent's (expected) utility maximization problem.

(d) Let x_1 be the risk-free asset demand. Show that $x_1(a)$ is non-increasing if $a < 1$.

(e) How does x_1 depend on π?

Problem 5

(a) Consider an agent endowed with the utility function over wealth $u(w) = \beta w^2 + \gamma w$, where $\beta < 0$ and $\gamma > 0$ (and assume that $w < -\gamma/2\beta$). Show that for any gamble F, the expected utility of F depends only on its mean and variance.

(b) Show that if F and G are two probability distributions over wealth such that $\mathbb{E}[F] = \mathbb{E}[G]$ and $Var(F) \leq Var(G)$, we do not necessarily have that $F \succsim G$ for a risk-averse agent.

Problem 6

There is a lottery that pays $x + \varepsilon$ with probability 1/2 and $x - \varepsilon$ with probability 1/2. Compute and interpret the second derivative of this lottery's certainty equivalent with respect to ε when $\varepsilon \to 0$.

Problem 7

Tversky and Kahneman (1986) report the following experiment: 150 participants receive each a questionnaire asking him to make two choices, one from $\{a, b\}$ and the second from $\{c, d\}$, where each prospect is played independently (the number in brackets represent the percentage of participants choosing each option):

(i). A sure profit of \$240. [84%]

(ii). A lottery between a profit of \$1000 with probability 25% and \$0 with probability 75%. [16%]

(iii). A sure loss of \$750. [13%]

(iv). A lottery between a loss of \$1000 with probability 75% and \$0 with probability

25%. [87%]

The participants will receive the sum of the outcomes of the two lotteries he chooses. Seventy-three percent of participants chose the combination a and d.

(a) Are the participants' attitudes towards risk consistent between decisions $\{a, b\}$ and $\{c, d\}$?

(b) Construct the simple lotteries induced by the compound lotteries (a, d) and (b, c). Can we rank them in first- or second-order stochastic dominance sense?

Problem 8

Consider an agent who can either invest all of his wealth ($w = 1$) in a risky asset g with expected return $\mathbb{E}[g] = 1$, or invest only a fixed proportion $\alpha \in (0, 1)$ in the risky asset while the remainder he invests in a risk-free asset (whose return is one). Show that if the agent is risk-averse, he prefers the second option to the first one.

(Hint: first, compute the expected return for the second option; and then use first- and second-order stochastic dominance results to support the claim.)

Exercises

Solve exercises 6.B.3, 6.B.5, 6.C.1 and 6.D.2 from Mas-Colell, Whinston and Green.