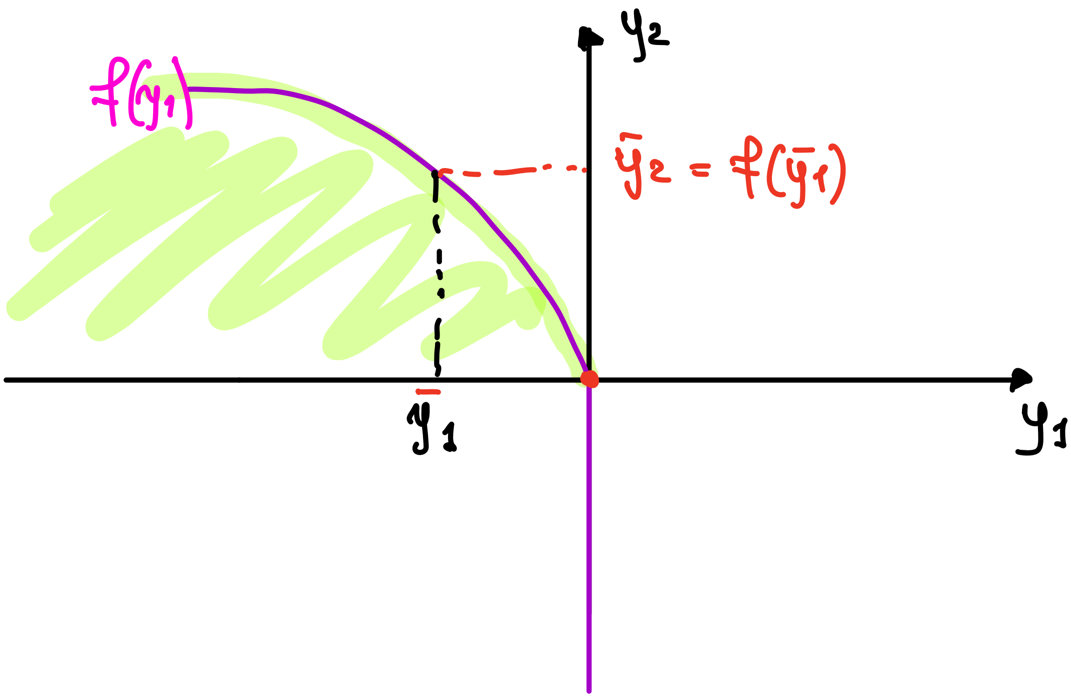
PRODUCTION SET & PRODUCTION FUNCTION

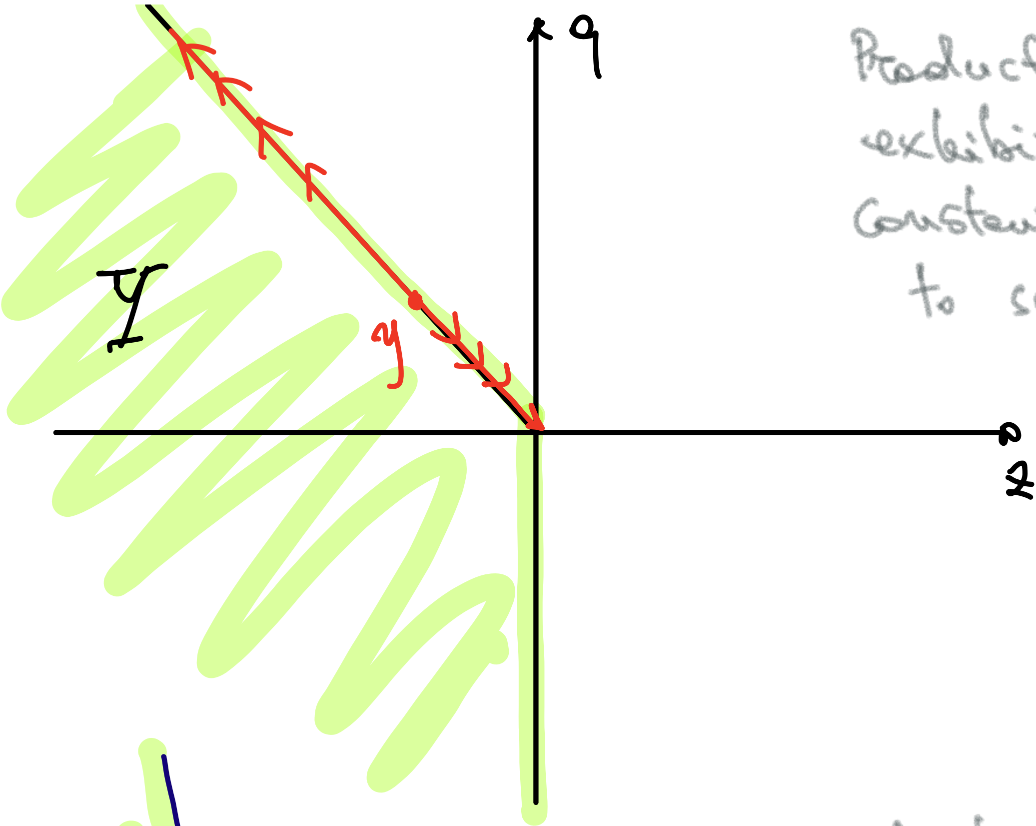
$m = 2$ $\qquad y_1 , y_2$

y_1 ↑ input $\qquad y_2$ ↑ output

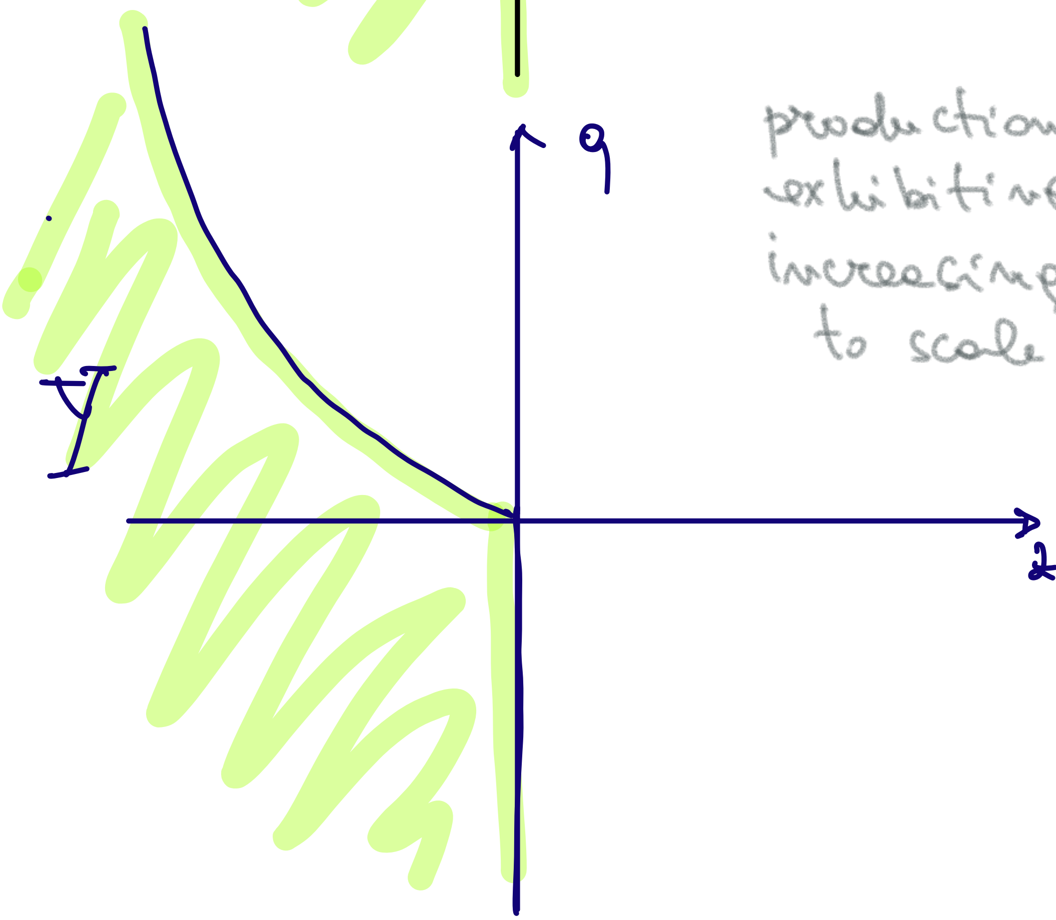


$$Y = \{ (y_1, y_2) \in \mathbb{R}^2 : \; y_2 - f(y_1) \leq 0 \}$$

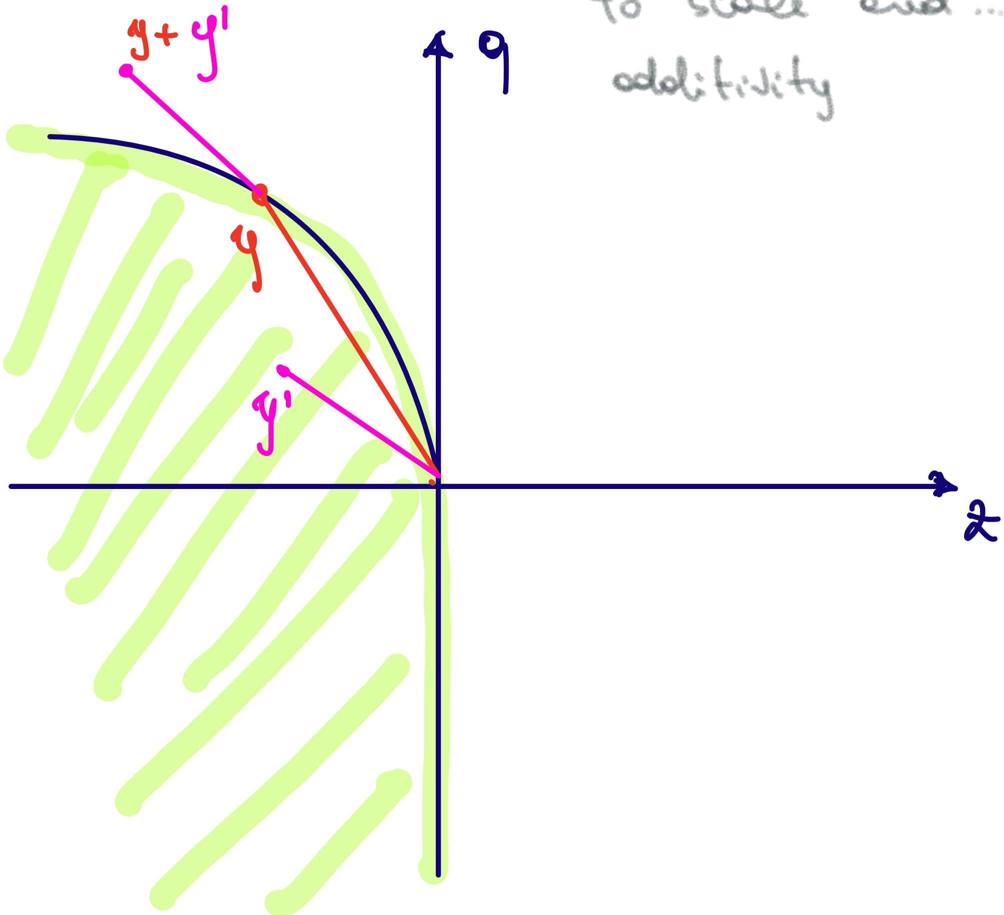
or $\quad Y = \{ (-z, q) \in \mathbb{R}^2 : \; q - f(z) \leq 0 \}$



Production set exhibiting constant returns to scale



production set exhibiting increasing returns to scale production set with decreasing returns to scale and ... additivity



PROFIT MAXIMIZATION PROBLEM

$$\max_{q, z \in \mathbb{R}^n} \quad p \cdot q - w_1 z_1 - \dots - w_{n-1} z_{n-1}$$
$$\text{s.t.} \quad q - f(z_1, \dots, z_{n-1}) \leq 0$$
$$z_i \geq 0 \quad \forall\, i = 1, \dots, n-1$$

By direct substitution, we can solve

$$\max_{z_1, \dots, z_{n-1}} \quad p \cdot f(z_1, \dots, z_{n-1}) - \sum_{i=1}^{n-1} w_i z_i$$

FOC.
$$p \cdot \frac{\partial f(z)}{\partial z_i} - w_i \leq 0 \qquad \forall\, i = 1, \dots, n-1$$

with strict equality if $z_i > 0$

Optimality

$$p \cdot MP_i = w_i \qquad \forall\, i$$

value of mg product equal to factor price

$\forall$ pair of inputs, say ℓ and K, we must have that

$$p \cdot \frac{\partial f(z)}{\partial z_\ell} = w_\ell \qquad z_\ell^* > 0$$

$$p \cdot \frac{\partial f(z)}{\partial z_K} = w_K \qquad z_K^* > 0$$

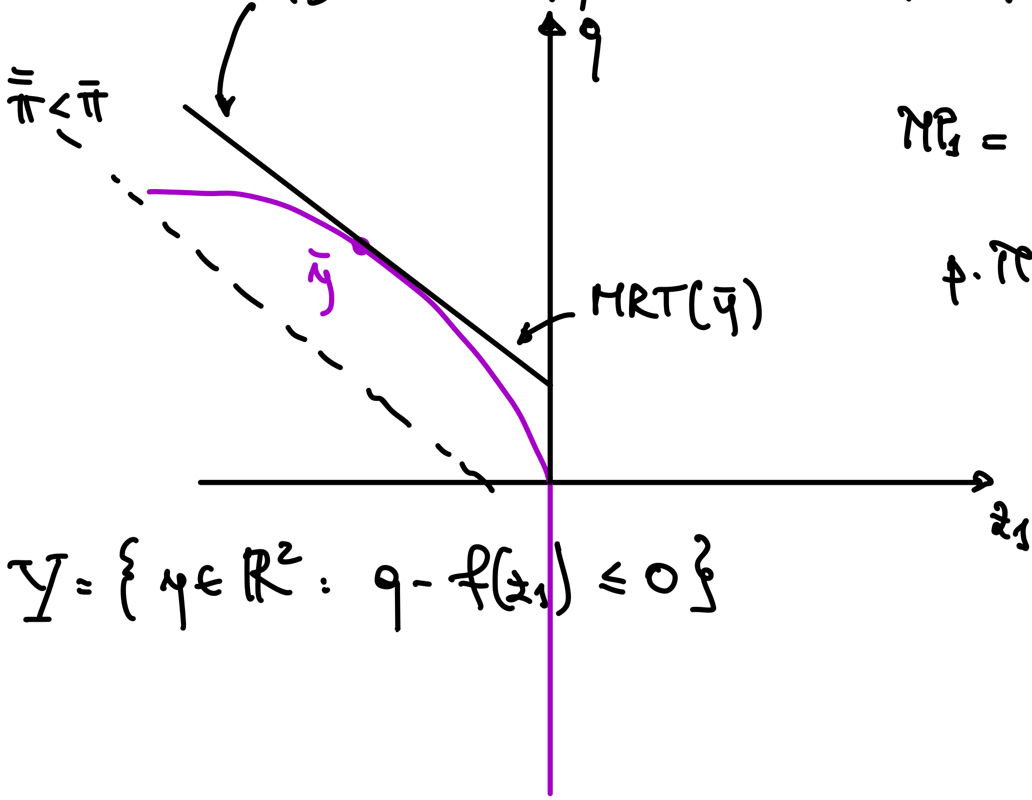
$$\Rightarrow \quad \frac{\dfrac{\partial f(z)}{\partial z_\ell}}{\dfrac{\partial f(z)}{\partial z_K}} = \frac{w_\ell}{w_K}$$

At the profit-max production plan the mg rate of technical substitution for (ℓ, K) is equal to the ratio of the prices of the two factors used in production.

THE PROFIT MAXIMIZATION PROBLEM

$m = 2 \qquad y = (q, z_1) \in \mathbb{R}^2$

iso-profit $\{ y \in \mathbb{R}^2 \text{ s.t. } pq - w_1 z_1 = \bar{\pi} \}$, slope $\frac{w_1}{p}$



$MP_1 = \frac{w_1}{p}$

$p \cdot MP_1 = w_1$

$Y = \{ y \in \mathbb{R}^2 : q - f(z_1) \le 0 \}$

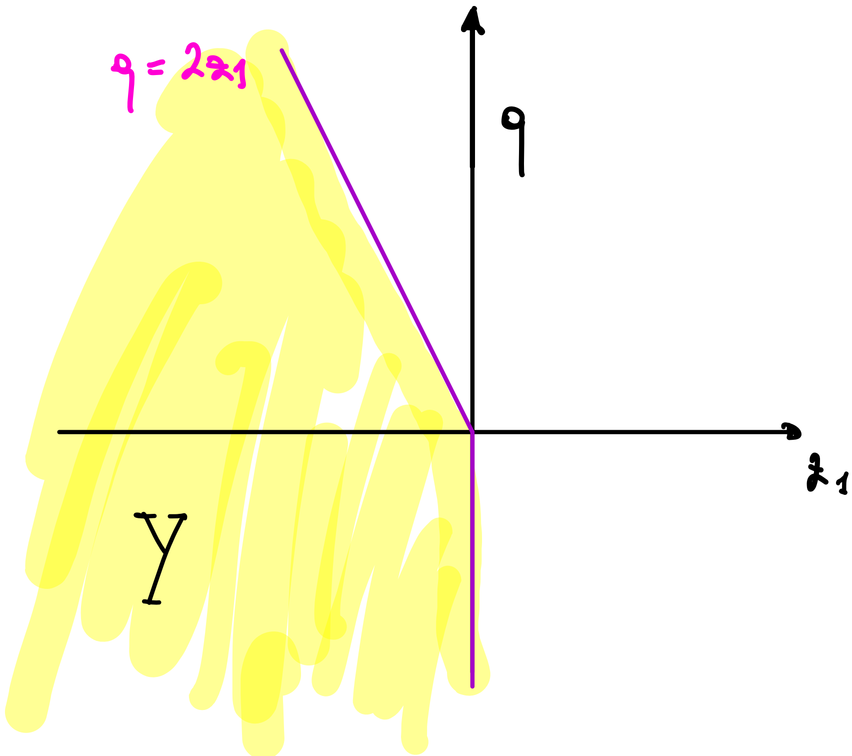
Slope of the iso-profit line equal to the slope of the prod. function at the optimal production plan.

When the production set exhibits constant or increasing returns to scale (i.e., non-decreasing returns to scale), then either profits are

$$\pi(p) = +\infty \quad \text{or} \quad \pi(p) = 0$$

PROFITS WITH CONSTANT or INCREASING RETURNS TO SCALE

$n = 2$



$f(z_1) = 2z_1$ CRS technology

$$\pi(p, w_1) = p \cdot f(z_1) - w_1 z_1 =$$
$$= p \cdot 2z_1 - w_1 z_1 =$$
$$= (2p - w_1)\, z_1$$

If $2p < w_1 \Rightarrow \nexists\, x_1 > 0$ that the firm is willing to demand

Hence, $\pi(p) = 0$

If $2p > w_1 \Rightarrow$ demand of input is unbounded $\Rightarrow \pi(p) = +\infty$

OBSERVATION ON PMP

Let $y \in y(p)$ be a supply correspondence at prices p and $y' \in y(p')$ solve PMP at prices p'.

By definition,

$$p \cdot y \geq p \cdot \tilde{y} \qquad \forall\, \tilde{y} \in Y \text{ including } y'$$

and

$$p' \cdot y' \geq p' \cdot \tilde{y} \qquad \forall\, \tilde{y} \in Y \text{ including } y$$

Specifically,

$$p \cdot y \geq p \cdot y'$$

and

$$p' \cdot y' \geq p' \cdot y$$

that is

$$pq - \sum_{i=1}^{m-1} w_i z_i \geq p \cdot q' - \sum_{i=1}^{m-1} w_i z_i' \qquad (A)$$

and

$$p' q' - \sum_i w_i' z_i' \geq p' q - \sum_i w_i' z_i \qquad (B)$$

Rearrange :

$$p(q-q') - \sum_{i=1}^{m-1} w_i (z_i - z_i') \geq 0 \qquad (A)$$

$$-\left[p'(q-q') - \sum_{i=1}^{m-1} w_i' (z_i - z_i')\right] \geq 0 \qquad (B)$$

(A)+(B) yields :

$$(p-p')(q-q') - \sum_{i=1}^{m-1} (w_i - w_i')(z_i - z_i') \geq 0$$

let $\Delta p \equiv p - p'$ $\quad \Delta q \equiv q - q'$ ... and so on

(A+B) reads as

$$\boxed{\Delta p \, \Delta q - \Delta w_1 \Delta z_1 - \dots - \Delta w_{m-1} \Delta z_{m-1} \geq 0}$$

if $\Delta p > 0 \quad \Delta w_1 = \Delta w_2 = \dots = \Delta w_{m-1} = 0$

$(A+B) \Rightarrow \Delta p \, \Delta q \geq 0 \qquad \Delta p > 0 \;\; \Delta q > 0$

if $\Delta p = 0 \quad \Delta w_1 > 0 \quad \Delta w_2 = \dots = \Delta w_{m-1} = 0$

$(A+B) \Rightarrow \Delta w_1 \Delta z_1 \leq 0 \qquad$ if $\Delta w_1 > 0$

$\Delta z_1 \leq 0$

the law of supply is established via profit maximisation

COST MINIMISATION PROBLEM

Fix q, and search for the vector $(z_1, \dots, z_{m-1})$ s.t. the cost of producing q is minimised

$$\max_{z_1, \dots, z_{m-1}} \; -(w_1 z_1 + w_2 z_2 + \dots + w_{m-1} z_{m-1})$$

$$\text{s.t.} \quad q - f(z_1, z_2, \dots, z_m) \leq 0$$

$$z_i \geq 0 \qquad \forall\, i = 1, \dots, m-1$$

Use the Lagrangian method

$$\mathcal{L}(z_1, \dots, z_{m-1}, \lambda) = -(w_1 z_1 + \dots + w_{m-1} z_{m-1}) + \lambda\left[f(z_1, z_2, \dots, z_{m-1}) - q\right]$$

then solve the system of equations:

$$\frac{\partial \mathcal{L}}{\partial z_i} \leq 0 \qquad z_i \geq 0 \qquad z_i \cdot \frac{\partial \mathcal{L}}{\partial z_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \geq 0 \qquad \lambda \geq 0 \qquad \lambda \cdot \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Specifically,

$$\frac{\partial \mathcal{L}}{\partial z_i} = -w_i + \lambda \frac{\partial f(z)}{\partial z_i} \leq 0 \qquad z_i \geq 0 \quad \forall i$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f(z) - q \geq 0 \qquad \lambda > 0$$

$$\forall \; z_i^* > 0 \qquad w_i = \lambda \frac{\partial f(z)}{\partial z_i}$$

take two inputs ℓ, k, optimality implies that

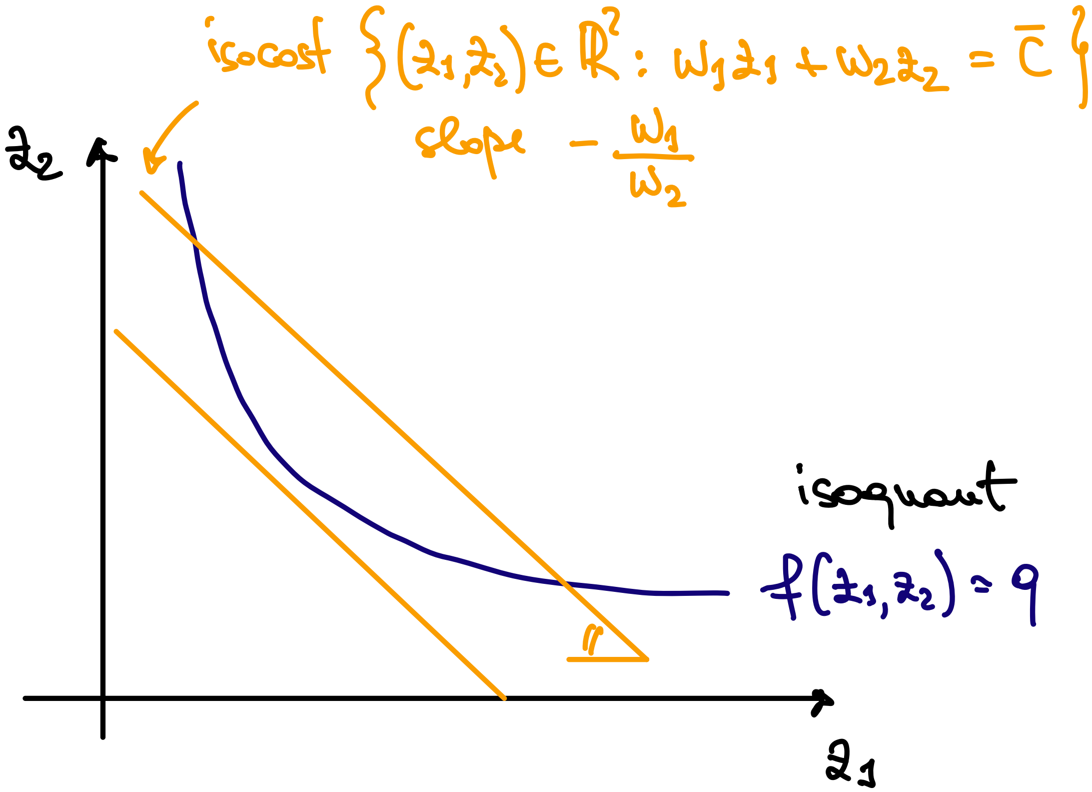
$$\frac{w_\ell}{w_k} = \frac{\partial f(z)/\partial z_\ell}{\partial f(z)/\partial z_k}$$

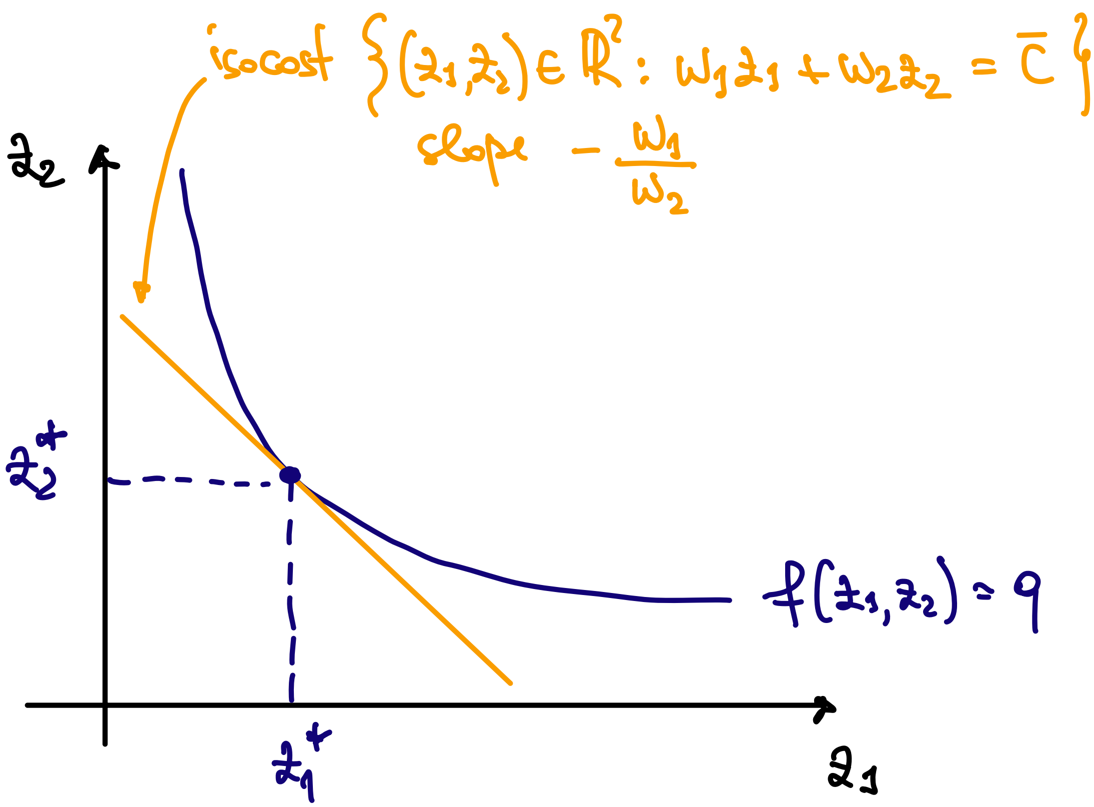
$MRTS_{\ell,k}$ = ratio between the prices of the two inputs ℓ, k

$\downarrow$

slope of the isoquant corresponding to the fixed level of production q

take a convex $Y \iff f(\cdot)$ concave, the isoquant is convex.





(z_1^*, z_2^*) are the conditional factor demands for inputs 1, 2.

$C(w_1,w_2,q) = w_1 z_1^* + w_2 z_2^*$ is the cost function

↳ then the firm chooses the q that max her profits $pq - C(w_1,w_2,q)$

LAGRANGE MULTIPLIER OF CMP

λ is the mg cost of production

$$c(w,q) = w \cdot z(w,q)$$

$$\frac{\partial c(w,q)}{\partial q} = w \cdot D_q z(w,q) \qquad (1)$$

From CMP, at the optimum the FOCs are

$$w_k = \lambda \frac{\partial f(z)}{\partial z_k} \qquad \forall k = 1, \dots, n-1$$

In matrix notation,

$$w = \lambda D_z f(z) \qquad (2)$$

Put (2) into (1), to get

$$\frac{\partial C(w,q)}{\partial q} = \lambda \, D_z f(z(w,q)) \cdot D_q z(w,q)$$

At the optimum, the constraint binds

$$f(z(w,q)) = q \qquad (*)$$

differentiating (*) w.r.t. q yields:

$$D_z f(z(w,q)) \cdot D_q z(w,q) = 1$$

Hence,

$$\frac{\partial C(w,q)}{\partial q} = \lambda$$

the Lagrange multiplier measures the mg effect of changing q on the cost function, i.e. the mg cost.

ON COST FUNCTIONS

Notice that $C(w,q)$ is a linear function of the input demands, hence whenever we consider the impact of an equally proportional change of all inputs on costs ... this is always of the same proportion of the change in inputs.

What we are interested in is the relationship between the cost function $C(w,q)$ and production q.

This depends on $f(\cdot)$.

Let $f(\cdot)$ have constant returns to scale

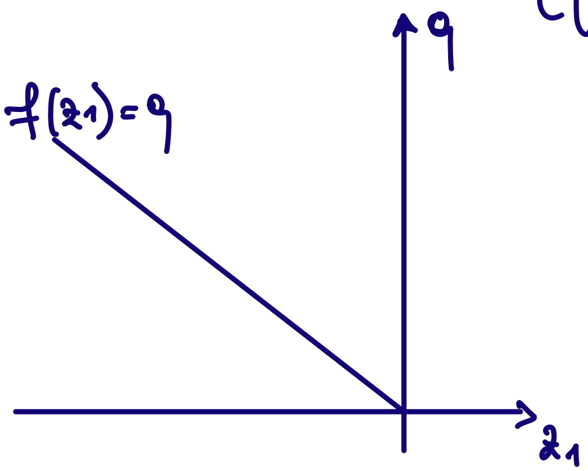
$$f(tz_1, \dots, tz_{M-1}) = t\, f(z_1, \dots, z_{M-1}) = tq.$$

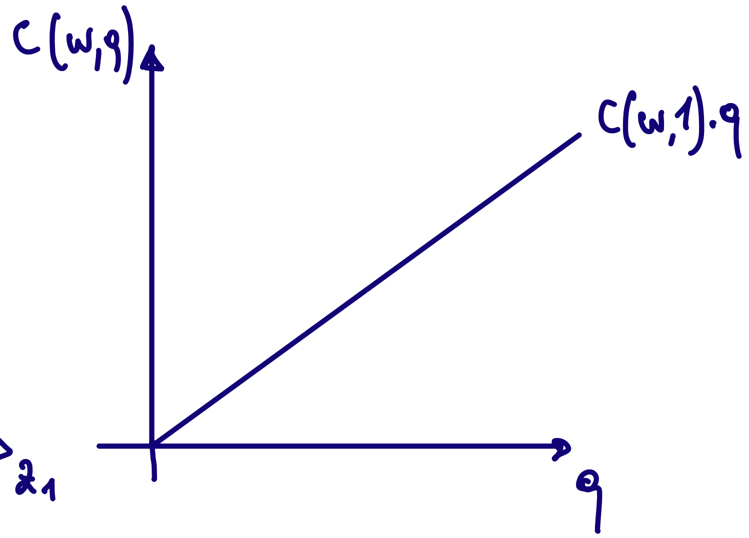
The minimal cost to produce 1 unit of output is $C(w,1)$.

What's the cost to produce q units?

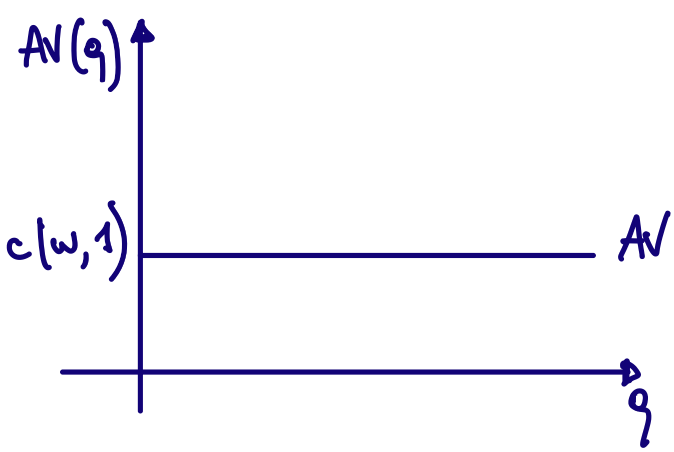
By CRS technology, the total cost is $c(w,1)\cdot q$, hence linear in q.

$$c(w,q) = c(w,1)\cdot q$$



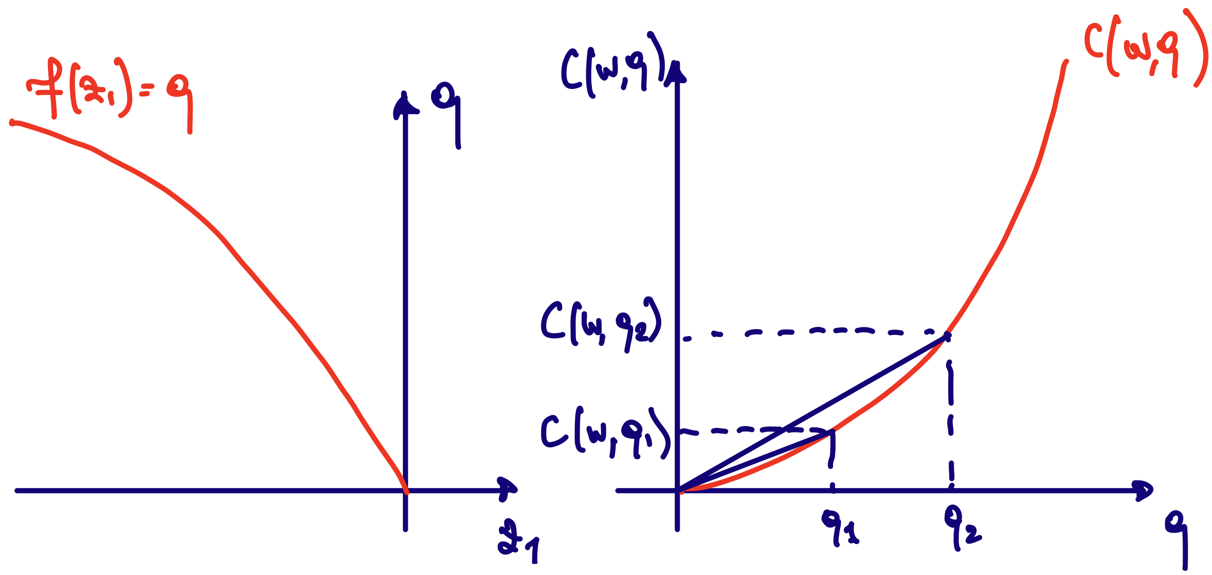


$$AV(q) = \frac{c(w,q)}{q} = c(w,1) \quad \text{constant}$$



What if the technology has decreasing returns to scale?

then if the firm wants to produce $2q$ she needs to more-than-double the purchase of inputs $\Rightarrow$ the total costs will more than dou and the average cost will be an increasing function.



think about an increasing returns to scale $f(\cdot)$!!