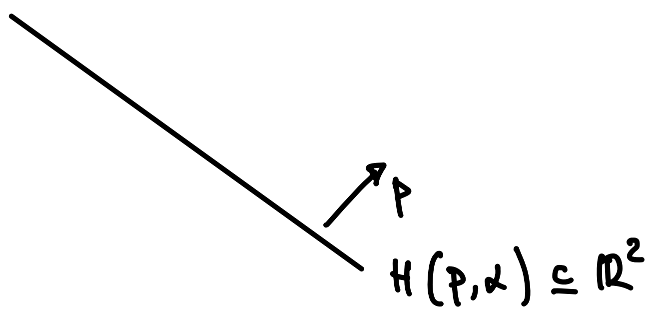
## HYPERPLANE in $\mathbb{R}^L$

the set of solutions of one linear equation in $L$ variables is called an hyperplane in $\mathbb{R}^L$, that is

$\forall\, p \in \mathbb{R}^L,\ p \neq 0$ and $\forall\, \alpha \in \mathbb{R}$ the set

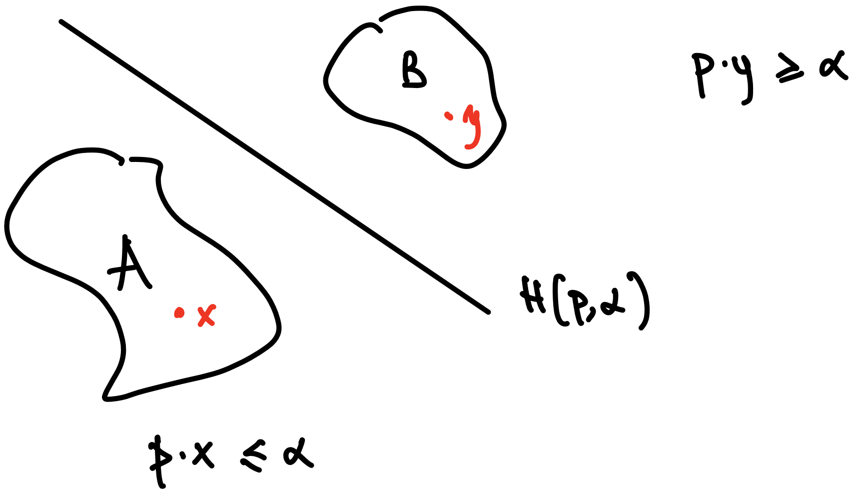
$$H(p,\alpha) = \{ x \in \mathbb{R}^L : \ p_1 x_1 + p_2 x_2 + \dots + p_L x_L = \alpha \}$$

is a hyperplane. The vector $p$ is called a normal to the hyperplane $H(p,\alpha)$



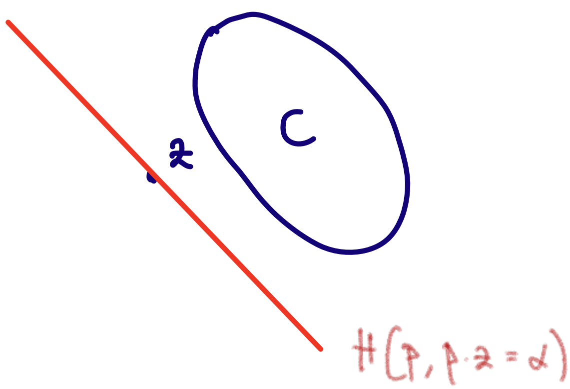

We say that $H(p,\alpha) \subset \mathbb{R}^2$ separates the two sets $A$ and $B$ in $\mathbb{R}^L$ if

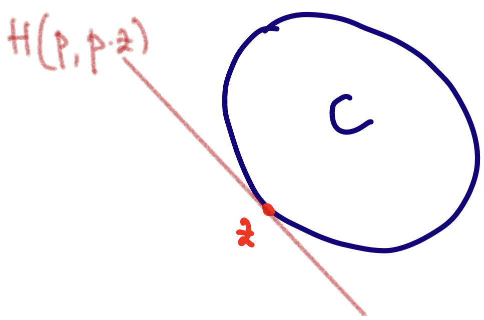
$$p \cdot x \leq \alpha \leq p \cdot y \qquad \forall\, x \in A,\ \forall\, y \in B$$




**Prop.** Let $C$ be a convex set of $\mathbb{R}^L$ and $z \notin C$. Then, there exists an hyperplane $H(p,\alpha)$ through $z$ which separates $z$ and $C$, i.e.

$$p \cdot z = \alpha \leq p \cdot x \qquad \forall\, x \in C$$




$z \in \overline{C}$

$\uparrow$

closure

Proof. Let $z \in \bar{C}$, with $\bar{C}$ being the closure of $C$.
There exists a sequence $(z_n^*)$ converging to $z$
with $z_n \notin \bar{C}$ $n = 1, \dots$
Hence, there exist normals $p_n \in \mathbb{R}^L$, $p_n \neq 0$
such that:

$$p_n \cdot z_n < p \cdot x \qquad \forall x \in \bar{C} \quad \text{with } n = 1, \dots$$

Let $q_n = \dfrac{p_n}{|p_n|}$ with $n = 1, \dots$

The sequence $q_n$ is bounded (by 0 and 1) and admits a converging sub-sequence, say $q_{n_t} \xrightarrow[t]{} p$.
$p \neq 0$ given $q_n$.
Since the $\cdot$ product is a continuous function

$$p \cdot z \leq p \cdot x \quad \forall x \in \bar{C}$$

Hence, $H(p, p \cdot z)$ is the desired hyperplane.