

Microeconomics I - General Equilibrium

December 9, 2024

- We now focus on the problems of organization of production and of the allocation of commodities to consumers.
- From a positive perspective, investigate how different institutional arrangements affect production and consumption;
- From a normative perspective, investigate which production and consumption allocations are social optima.
- The institutional setting is a market economy with private ownership. Consumers privately own assets/goods which then trade in the market-place for other assets/goods. Firms decide on production plans, acquire inputs and sell outputs ... and are owned by consumers.

Introduction: competitive market economies

- In a competitive market economy, every commodity is traded in a market at publicly known prices and all agents act as price takers.
- We will use the notions of competitive (or Walrasian) equilibrium and of Pareto optimality (or Pareto efficiency).
- We will discuss the two Fundamental Theorems of Welfare Economics

Introduction: First and Second Welfare Theorems

- **First Theorem of Welfare Economics:** If every relevant good is traded in a market at publicly known prices (i.e. the set of markets is complete) and if households and firms act perfectly competitively (i.e. take market prices as given) then the market outcome is Pareto optimal. When markets are complete, any competitive equilibrium is Pareto optimal.
- **Second Theorem of Welfare Economics:** If consumers' preferences and firms' production sets are convex and if there is a complete set of markets with publicly known prices, and if every agent acts as a price taker, then any Pareto optimum can be achieved as a competitive equilibrium with appropriate lump-sum transfers of wealth.

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Introduction: First and Second Welfare Theorems

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- ... if the decisions of one agent directly affect the utility or the production function of other agents, externalities emerge and markets' completeness is violated...
- ... if some agents have market power, the price-taking assumption is violated ...
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Introduction

- We will examine competitive market economies from a *general equilibrium* perspective.
- Its most striking feature is that it views the economy as a *closed* and *interrelated* system.
- More precisely, we must simultaneously determine the equilibrium values of all variables of interest.
- The exogenous variables are kept to a minimum: the set of economic agents, the available technologies, the preferences and the physical endowments of goods of the various agents.

- This perspective stands in contrast to the *partial equilibrium* approach we have taken so far.
- Partial equilibrium analysis disregards those endogenous variables not directly related to the problem at hand (and transforms them into exogenous vars), now any perturbation in the economic environment leads to a revaluation of all endogenous economic variables.
- In this sense, general equilibrium has a more specific meaning: it is a theory of the determination of equilibrium prices and quantities in a *system* of related markets.

- Our goal is twofold:
 - First, we will define two central concepts: *Pareto optimality* and *Walrasian equilibrium*.
 - Secondly, we prove two results: the *First and Second Fundamental Welfare Theorems*.
- Examining the relationship between Pareto optimality and Walrasian equilibrium is the basis for the derivation of the Fundamental Welfare Theorems.

Exchange Economies

Notations and Definitions

- There is a finite number I of agents and L of goods.
- The consumption set of each agent $i \in \mathcal{I} = (1, \dots, I)$ is \mathbb{R}_+^L .
- We denote the profile of consumptions of agent i with $x_i = (x_{1i}, \dots, x_{Li}) \in \mathbb{R}_+^L$,
in which x_{ki} denotes the amount of good k consumed by agent i .

Definition

An **exchange economy** is a list $\mathcal{E} = \{u_i, \omega_i\}_{i \in \mathcal{I}}$, where:

- i) $\mathcal{I} = \{1, \dots, I\}$ is a finite set of agents;
- ii) $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}$ is agent i 's utility function;
- iii) $\omega_i \in \mathbb{R}_+^L$ is agent i 's endowment, with $\omega_i = (\omega_{1i}, \dots, \omega_{Li})$.

- Denote by $\omega = \sum_{i \in \mathcal{I}} \omega_i$ the aggregate endowment vector.
- Notice that $\omega = (\sum_{i \in \mathcal{I}} \omega_{1i}, \dots, \sum_{i \in \mathcal{I}} \omega_{Li})$.
- Assume that $\omega \gg 0$.

In other words, we assume there is a positive amount of each good in the economy.

Exchange Economies

Allocations

- An *allocation* is a list $x = (x_1, \dots, x_I)$ where $x_i \in \mathbb{R}_+^L$ is a consumption vector for an agent i .
- An allocation $x = (x_1, \dots, x_I)$ is *feasible* if

$$\sum_{i \in \mathcal{I}} x_{li} \leq \sum_{i \in \mathcal{I}} \omega_{li} \quad \text{for } l = 1, \dots, L.$$

- The aggregate consumption of every good cannot exceed the available aggregate endowment. Feasibility admits free disposal of goods.
- Note that the set of feasible allocations only depends on the aggregate endowment vector $\omega = \sum_{i \in \mathcal{I}} \omega_i$.

Exchange Economies

Pareto Efficiency

Definition

A feasible allocation \bar{x} is **Pareto efficient/optimal** if there exists no feasible allocation x such that $u_i(x_i) \geq u_i(\bar{x}_i)$ for all $i = 1, 2, \dots, I$ and $u_i(x_i) > u_i(\bar{x}_i)$ for some i .

Walrasian Equilibrium

Definition

Let $\mathcal{E} = \{u_i, \omega_i\}_{i \in \mathcal{I}}$ be an exchange economy, $p^* \gg 0$ a vector of prices, and x^* as allocation. The pair (p^*, x^*) is a **Walrasian equilibrium** if:

i) for each $i \in \mathcal{I}$, x_i^* solves

$$\begin{aligned} \max \quad & u_i(x_i) \\ \text{s.t.} \quad & p^* \cdot x_i \leq p^* \cdot \omega_i \\ & x_i \in \mathbb{R}_+^L; \end{aligned}$$

ii) for each $l = 1, \dots, L$,

$$\sum_{i \in \mathcal{I}} x_{li}^* = \sum_{i \in \mathcal{I}} \omega_{li}.$$

Walrasian Equilibrium

- In other words, a Walrasian equilibrium identifies a vector of prices p^* and an allocation x^* such that:
 - x^* is feasible and
 - each consumer maximizes his utility given the price vector p^* .
- An allocation x^* is a *Walrasian allocation* if there exists $p^* \gg 0$ that sustains (p^*, x^*) as a Walrasian equilibrium.
- Notice that the set of Walrasian allocations depends on the initial endowments $(\omega_1, \dots, \omega_I)$ and on the preferences of each agent.
- The set of Walrasian allocations is denoted $W(\mathcal{E})$.

Walrasian Equilibrium

Some Remarks

- We impose two (important) behavioral assumptions on the definition of a Walrasian equilibrium.
 - ① *Competitive hypothesis*: all agents take prices as given.
 - ② Given a price vector, agents believe they can buy and sell as much as they want of each good, as long as their budget constraint is satisfied.
- At this stage, we will remain silent about the formation of prices.

- Let us now consider an exchange economy \mathcal{E} with $I = L = 2$ and no production.
- Pure exchange model (Edgeworth Box).
- Let us now recover the notions of an allocation, of a feasible allocation, of Pareto optimality and a competitive equilibrium in the Edgeworth Box.

See the Note on GE-Edgeworth Box 2024

Existence of a Walrasian equilibrium

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Walrasian Equilibrium

Existence of Walrasian Equilibrium

- One question we need to address is the existence of a Walrasian equilibrium.
- We do so by relying on our knowledge of consumer theory.
- Consider an exchange economy $\mathcal{E} = \{u_i, \omega_i\}_{i \in \mathcal{I}}$ where each u_i represents preferences that are continuous, locally non-satiated and strictly convex.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

- For every vector of prices $p \gg 0$ and every wealth $w \geq 0$, consumer i 's problem is given

$$\begin{array}{ll} \max & u_i(x_i) \\ \text{s.t.} & p \cdot x_i \leq w \\ & x_i \in \mathbb{R}_+^L \end{array}$$

- Given the assumptions about u_i , this problem has a unique solution labelled as $x_i(p, w)$.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

- Because a consumer i 's wealth w_i is given by his initial endowment in an exchange economy, we have that $w_i = p \cdot \omega_i$
- Write $x_i(p, p \cdot \omega_i)$. By construction, $\sum_{i=1}^I x_{ki}(p, p \cdot \omega_i)$ is the aggregate demand of good k when the vector of prices is p .
- Let

$$z_k(p) = \sum_{i=1}^I [x_{ki}(p, p \cdot \omega_i) - \omega_{ki}] ,$$

then, $z_k(p)$ is the aggregate excess demand of good k when the vector of prices is p .

Walrasian Equilibrium

Existence of Walrasian Equilibrium

- Given the excess demand for good k

$$z_k(p) = \sum_{i=1}^I [x_{ki}(p, p \cdot \omega_i) - \omega_{ki}] ,$$

let $z(p) = (z_1(p), \dots, z_L(p))$ be the vector of excess demands of all goods, with $z : \mathbb{R}_{++}^L \rightarrow \mathbb{R}^L$. Notice that

$$z(p) = \sum_{i=1}^I [x_i(p, p \cdot \omega_i) - \omega_i] .$$

Walrasian Equilibrium

Existence of Walrasian Equilibrium

Lemma

The pair (p^*, x^*) is a Walrasian equilibrium if, and only if, $x_i^* = x_i(p^*, p^* \cdot \omega_i)$ for each $i = 1, \dots, I$ and $z(p^*) = 0$.

Walrasian Equilibrium

Proof of Lemma

• (\Rightarrow)

- Suppose (p^*, x^*) is a Walrasian equilibrium.
- By assumption, $x_i^* = x_i(p^*, p^* \cdot \omega_i)$ for all $i \in \mathcal{I}$ and

$$\sum_{i=1}^I x_i^* = \sum_{i=1}^I \omega_i \Leftrightarrow \sum_{i=1}^I [x_i^* - \omega_i] = z(p^*) = 0.$$

• (\Leftarrow)

- Suppose now that (p^*, x^*) is such that $x_i^* = x_i(p^*, p^* \cdot \omega_i)$ for all $i \in \mathcal{I}$ and $z(p^*) = 0$.
- Then, (p^*, x^*) is a Walrasian equilibrium.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

- The previous result shows that the exchange economy \mathcal{E} has a Walrasian equilibrium if, and only if, there exists $p^* \gg 0$ such that $z(p^*) = 0$.
- Thus, if we can determine conditions under which the equation $z(p) = 0$ has a solution in the space of strictly positive prices \mathbb{R}_{++}^L , we have conditions under which \mathcal{E} has a Walrasian equilibrium.
- The next result lists properties of the excess demand function, which guarantee existence of Walrasian equilibria.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

Lemma - ExcessD

Assume that for every consumer i , $X_i = \mathbb{R}_+^L$ and \succsim_i are continuous, strictly convex and strongly monotone. Suppose that $\omega \gg 0$. Then, the function z has the following properties:

- i) z is continuous;
- ii) z is homogenous of degree zero;
- iii) $p \cdot z(p) = 0$ for all $p \gg 0$ (Walras' Law);
- iv) there is an $s > 0$ such that $z_l(p) > -s$ for every commodity l and price p ;
- v) if $\{p^m\}$ is a sequence of prices that converges to p with $p \neq 0$ and $p_l = 0$ for some l , then $\max\{z_1(p^m), \dots, z_L(p^m)\} \rightarrow +\infty$.

Walrasian Equilibrium

Proof of Lemma - ExcessD

- The continuity of z follows from the continuity of the demand function $x^i(p, w)$ on p and w of every consumer.
- Since the budget set of agent i 's consumer problem with $w = p \cdot \omega_i$ does not change when we multiply all prices by the same positive constant, we have that $x^i(p, p \cdot \omega_i) = x^i(\alpha p, \alpha p \cdot \omega_i)$ for all $\alpha > 0$.
- Thus, $z(\alpha p) = z(p)$ for all $\alpha > 0$, i.e. z is homogenous of degree zero.

Walrasian Equilibrium

Proof of Lemma - ExcessD

- We now prove (iii).
- For this, notice that u_i strictly increasing implies that $p \cdot x_i(p, p \cdot \omega_i) = p \cdot \omega_i$ for all $i \in \mathcal{I}$. Thus,

$$\begin{aligned} p \cdot z(p) &= \sum_{k=1}^L p_k z_k(p) \\ &= \sum_{k=1}^L \sum_{i=1}^I p_k [x_{ki}(p, p \cdot \omega_i) - \omega_{ki}] \\ &= \sum_{i=1}^I \sum_{k=1}^L p_k [x_{ki}(p, p \cdot \omega_i) - \omega_{ki}] \\ &= \sum_{i=1}^I [p \cdot x_i(p, p \cdot \omega_i) - p \cdot \omega_i] = 0. \blacksquare \end{aligned}$$

Walrasian Equilibrium

Existence of Walrasian Equilibrium

Theorem - E

Let $z : \mathbb{R}_{++}^L \rightarrow \mathbb{R}$ be a function satisfying properties (i) to (v) in Lemma - ExcessD. Then, there exists $p^* \gg 0$ such that $z(p^*) = 0$.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

- Theorem E implies the existence of a Walrasian equilibrium.
- The conditions for existence boil down to properties of the consumers' utility functions.
- In particular, it is required that the utility functions be continuous, strictly increasing and strictly quasi-concave.
- As a corollary of Theorem - E, we can establish that under such conditions for the utility functions, the set $W(\mathcal{E})$ of an exchange economy \mathcal{E} is not empty.

Walrasian Equilibrium

Existence of Walrasian Equilibrium

Corollary - E

Let $\mathcal{E} = \{u_i, \omega_i\}_{i \in \mathcal{I}}$ with $\sum_{i \in \mathcal{I}} \omega_i \gg 0$ be such that the functions u_i are continuous, strictly quasi-concave, and strictly increasing. Then, \mathcal{E} has a Walrasian equilibrium.

General Equilibrium Economies with Production

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Production Economies

Notations and Definitions

- Let I be the finite number of consumers ($i = 1, \dots, I$); J be the finite number of firms ($j = 1, \dots, J$); and L be the finite number of goods ($l = 1, \dots, L$).
- The consumption set of every consumer i is \mathbb{R}_+^L , and her utility is $u_i(x_i)$.
- The production set of every firm j is $Y_j \subset \mathbb{R}^L$, with production plans $y_j = (y_{1j}, y_{2j}, \dots, y_{Lj}) \in \mathbb{R}^L$.

Production Economies

Endowment

- Let ω_i be consumer i 's vector of initial endowments, and let θ_{ij} be the share of profits of firm j to which consumer i is entitled.
- Since there is production, the total amount of good l available to the economy is $\sum_{i \in \mathcal{I}} \omega_{li} + \sum_{j \in \mathcal{J}} y_{lj}$

Production Economies

Allocations

- An *allocation* is a list $(x_1, \dots, x_I, y_1, \dots, y_J)$ that describes a consumption vector $x_i \in \mathbb{R}_+^L$ for every consumer i , and a production plan $y_j \in \mathbb{R}^L$ for every firm j .
- An allocation $(x_1, \dots, x_I, y_1, \dots, y_J)$ is *feasible* if

$$\sum_{i=1}^I x_{li} \leq \sum_{i=1}^I \omega_{li} + \sum_{j=1}^J y_{lj} \quad \text{for } l = 1, \dots, L.$$

Economies with production

Pareto Efficiency

Definition

A feasible allocation $(x_1, \dots, x_I, y_1, \dots, y_J)$ is **Pareto efficient/optimal** if there exists no feasible allocation $(\bar{x}_1, \dots, \bar{x}_I, \bar{y}_1, \dots, \bar{y}_J)$ such that $u_i(\bar{x}_i) \geq u_i(x_i)$ for all $i = 1, 2, \dots, I$ and $u_i(\bar{x}_i) > u_i(x_i)$ for some i .

Walrasian Equilibrium

The allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and the price vector $p^* \gg 0$ are a Walrasian equilibrium if:

- **Profit max:** for each firm j , y_j^* solves

$$\max p^* \cdot y_j \text{ with } y_j \in Y_j;$$

- **Utility max:** for each consumer i , x_i^* solves

$$\begin{aligned} \max \quad & u_i(x_i) \\ \text{s.t.} \quad & p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_{j=1}^J \theta_{ij}(p^* \cdot y_j^*) \\ & x_i \in \mathbb{R}_+^L; \end{aligned}$$

- **Market Clearing:** for each good l

$$\sum_{i=1}^I x_{li}^* = \sum_{i=1}^I \omega_{li} + \sum_{j=1}^J y_{lj}^*.$$

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- **Market Clearing:** for each good l

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Walrasian Equilibrium

- Observe that if $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*, p^*)$ are a Walrasian equilibrium, then so are the allocation $(x_1^*, \dots, x_I^*, y_1^*, \dots, y_J^*)$ and the price vector αp^* with $\alpha > 0$.
- Indeed, we normalize prices by setting one price equal to 1.

Lemma GE-2

If the allocation $(x_1, \dots, x_I, y_1, \dots, y_J)$ and the price vector $p \gg 0$ satisfy the market clearing condition in *iii*) for all goods $l \neq k$ and if every consumer's budget constraint holds as an equality, i.e. for every i ,

$$p \cdot x_i = p \cdot \omega_i + \sum_{j=1}^J \theta_{ij}(p \cdot y_j),$$

the market for good k also clears.

Walras law

Proof of Lemma GE-2

- Take all the budget constraints of all consumers and add them up, then separate good k from all other goods, you are left with:

$$\sum_{l \neq k} p_l \left[\sum_{i=1}^I x_{li} - \sum_{i=1}^I \omega_{li} - \sum_{j=1}^J y_{lj} \right] = p_k \left[\sum_{i=1}^I x_{ki} - \sum_{i=1}^I \omega_{ki} - \sum_{j=1}^J y_{kj} \right]$$

since $\sum_{i=1}^I \theta_{ij} = 1$.

- Since market clearing condition holds for all $l \neq k$ goods, the left-hand side of the expression is equal to zero.
- Then, also the right-hand side is equal to zero. Since $p_k > 0$, the market for good k clears, too. ■

First Welfare Theorem

Price Equilibrium with Transfers

Refer to an economy with production as $\mathcal{E}^P = \left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega} \right)$.

Definition

In the economy \mathcal{E}^P , an allocation (x^*, y^*) and a price vector $p^* = (p_1^*, \dots, p_L^*)$ constitute a *price equilibrium with transfers* if there exists a vector (w_1, \dots, w_I) , that is an assignment of wealth levels, with

$$\sum_{i=1}^I w_i = p^* \cdot \bar{\omega} + \sum_{j=1}^J p^* \cdot y_j^*$$

such that:

- ① $p^* \cdot y_j \leq p^* \cdot y_j^*, \quad \forall y_j \in Y_j$
- ② $x_i^* \succsim_i x_i, \quad \text{for every } \{x_i \in X_i : p \cdot x_i \leq w_i\}$
- ③ $\sum_{i=1}^I x_i^* = \bar{\omega} + \sum_{j=1}^J y_j^*.$

Price Equilibrium with Transfers

- A price equilibrium with transfers just requires that there is some way to distribute wealth among consumers so that the allocation and the prices constitute an equilibrium.
- A competitive equilibrium is a special case of price equilibrium with transfers, in which we assume a specific wealth distribution in the economy.

The First Fundamental Theorem of Welfare Economics

- We are now ready to examine the relationship between price equilibria and Pareto optimality.
- The First Welfare Theorem states conditions under which any price equilibrium with transfers (thus also any competitive equilibrium) is efficient.
- The requirement imposed on preferences is local non-satiation.

The First Fundamental Theorem of Welfare Economics

FW Theorem

If preferences are locally non-satiated for every consumer, and if (x^*, y^*, p^*) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal.

Proof of the First Welfare Theorem

- Assume (x^*, y^*, p^*) is a price equilibrium with transfers, with associated wealth levels such that $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$.

1. Utility maximization implies that

$$\text{if } x_i \succ_i x_i^* \implies p \cdot x_i > w_i$$

Bundles that are strictly preferred to x_i^* must be unaffordable.

- ## 2. In addition, LNS implies that $x_i \succsim_i x_i^* \implies p^* \cdot x_i \geq w_i$
- Bundles that are as least as good as x_i^* are at best just affordable.

- Suppose now by contradiction that there is an allocation (x, y) which Pareto dominates (x^*, y^*) . Then, $x_i \succsim_i x_i^* \forall i$ and $x_i \succ x_i^*$ for some i .
- By 1. and 2.,
$$\sum_i p^* \cdot x_i > \sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$$

Proof of the First Welfare Theorem

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Proof of First Welfare Theorem

- Profit maximization implies that

$$p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^* \geq p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j$$

$$\implies \sum_i p^* \cdot x_i > p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j.$$

- Market clearing, $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$, implies that there is no profile of feasible production plans that can generate enough social resources to match the total cost $\sum_i p^* \cdot x_i$. Thus, (x, y) is not feasible.
- Hence, (x^*, y^*) is undominated, hence Pareto optimal. ■

Proof of First Welfare Theorem

- Profit maximization implies that

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Discussion of First Welfare Theorem

- Think about an example which shows that if preferences are not LNS, then the result may not hold.
- As we mentioned before, the first welfare theorem is valid when agents are price takers and every commodity has a market with publicly known prices, i.e., markets are complete. Under these assumptions, competitive market allocations are Pareto efficient.
- Suppose there is a central institution (benevolent social planner) who has the objective to reach a certain efficient allocation of consumptions and productions. Let this planner impose redistribution of wealth, for example through lump-sum taxation. In this context, we can state the Second Theorem of Welfare Economics.

The Second Welfare Theorem

- The SWT gives us the conditions under which *any* Pareto efficient allocation can be supported at a price equilibrium with transfers.

Price Equilibrium and Quasi-equilibrium

- Recall that an allocation (x^*, y^*) and a price vector p^* constitute a *price equilibrium with transfers* if there exists a profile of wealth levels (w_1, \dots, w_I) with $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$ such that:
 - profit maximization
 - x_i^* is maximal for \succsim_i in the budget set $\{x_i \in X_i : p^* \cdot x_i \leq w_i\}$
 - market clearing
- condition 2) can be equivalently stated as:

$$x_i \succ_i x_i^* \implies p^* \cdot x_i > w_i, \forall i, x_i^* \in \mathbb{R}_+^L, p^* \cdot x_i^* \leq w_i$$

Price Quasi-equilibrium with Transfers

Definition - QuasiE

Given an economy $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\right)$, an allocation (x^*, y^*) and a price vector p^* constitute a *price quasi-equilibrium with transfers* if there exists a vector of wealth levels (w_1, \dots, w_I) , with $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$ such that:

- 1 $p^* \cdot y_j \leq p^* \cdot y_j^* \quad \forall y_j \in Y_j$
- 2 $x_i \succsim_i x_i^* \implies p^* \cdot x_i \geq w_i, \quad \forall i, x_i^* \in \mathbb{R}_+^L, p^* \cdot x_i^* \leq w_i$
- 3 $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

In a quasi-equilibrium, bundles that are strictly preferred to x_i^* are at least as expensive as x_i^* . The idea is that the quasi-equilibrium bundle is the one that minimizes the consumer's expenditure to attain the max utility.

The Second Welfare Theorem

Proposition - SWT

Consider an economy \mathcal{E}^P . Suppose that every Y_j is convex, every preference relation \succsim_i is convex and LNS. Then, for every Pareto optimal allocation (x^*, y^*) , there is a price vector $p = (p_1, \dots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasi-equilibrium with transfers.

Proposition QE-E

Suppose that for every consumer, X_i is convex, $0 \in X_i$ and \succsim_i is continuous. Then, any price quasi-equilibrium with transfers that has $(w_i, \dots, w_I) \gg 0$ is a price equilibrium with transfers.

First-order conditions for Pareto Optimality

First-Order Conditions for Pareto Optimality

Additional Structure I

- We now consider an economy with additional assumptions about differentiability of utility and production functions.
- More precisely, go back to a pure-exchange economy such that:
 - 1) $X_1 = \dots = X_I = \mathbb{R}_+^L$,
 - 2) Preferences for each consumer are represented with an utility function $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}$, which is twice continuously differentiable and satisfy $\nabla u_i(x_i) \gg 0$ at all x_i (preferences are strongly monotone). We also normalize so that $u_i(0) = 0$.
- To characterise the set of Pareto optimal allocations we need to solve the following maximisation problem.

Maximisation problem 1

An allocation $x = (x_1, \dots, x_I)$ is Pareto optimal if it solves the following problem:

$$\text{Max} \quad u_1(x_{1i}, \dots, x_{Li})$$

$$s.t. \quad u_i(x_{1i}, \dots, x_{Li}) \geq \bar{u}_i, \quad i = 2, \dots, I \quad (1)$$

$$\sum_i x_{li} \leq \bar{\omega}_l, \quad l = 1, \dots, L \quad (2)$$

First-Order Conditions for Pareto Optimality

Pareto Optimality

- Given the assumptions on $u_i(\cdot)$, focus on non-negative utility levels $\bar{u}_i \geq 0$ for all i .
- Thus, finding a Pareto optimal allocation is equivalent to trying to maximize consumer 1's utility subject to
 - ① some minimal utility level for consumers $i \geq 2$,
 - ② the resource constraint in the economy.
- Moreover, the assumptions on $u_i(\cdot)$ imply that all the constraints (1)-(2) above are binding at a solution.

First-Order Conditions for Pareto Optimality

Kuhn-Tucker necessary conditions for max problem 1

- Let $(\delta_1, \dots, \delta_I) \geq 0$, $(\mu_1, \dots, \mu_n) \geq 0$ be the Lagrange multipliers associated with each constraint on Maximisation problem 1.
- Let $\delta_1 = 1$.
- We can thus write the Lagrangean

$$\max_x \quad \sum_{i=1}^I \delta_i u_i(x_i) + \sum_{l=1}^L \mu_l \left[\bar{\omega}_l - \sum_i x_{li} \right]$$

First-Order Conditions for Pareto Optimality

The Lagrangean's First-Order Conditions

- The (Kuhn-Tucker) necessary and sufficient first-order conditions are given by

$$x_{li} : \quad \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l \begin{cases} \leq 0 & \text{if } x_{li} = 0 \\ = 0 & \text{if } x_{li} > 0 \end{cases},$$

for all i, l .

First-Order Conditions for Pareto Optimality

Discussion of The Lagrangean's FOCs I

- Value of μ_l at optimum = increase of Mr 1's utility that would result from a marginal increase of $\bar{\omega}_l$. That is, μ_l measures the marginal value of good l .
- Value of δ_i at optimum = marginal change in consumer's i utility that would result from a decrease of the utility requirement \bar{u}_i for a consumer $i \neq 1$.
- Thus the condition on x_{li} , at an interior optimum, means that the marginal change of utility for Mr i from receiving more of good l , weighted by she is valued in terms of Mr 1's utility, should be equal to the marginal value of good l .

First-Order Conditions for Pareto Optimality

Ratio Conditions I

- Assuming an interior solution, the FOCs help characterizing the equilibrium of this economy.

Tangency between consumers i and h indifference curves:

$$\frac{\partial u_i / \partial x_{li}}{\partial u_i / \partial x_{l'i}} = \frac{\partial u_h / \partial x_{lh}}{\partial u_h / \partial x_{l'h}} \quad \text{for all } i, h, l, l'.$$

- This means that in any PO allocation, each consumer's MRS between any pair of goods must be equal (remember we saw this in the Edgeworth box, exchange economy).

First-Order Conditions for Pareto Optimality

The First Fundamental Welfare Theorem I

- Let us now relate the FOCs for PO with the fundamental welfare theorems.
- Assume that $u_i(\cdot)$ is quasi-concave for every i , i.e. preferences are convex.
- Then (x^*, p) is a price equilibrium with transfers with associated wealth levels $w_i = p \cdot x_i^*$ iff the first-order conditions for the constrained UMP for every i are satisfied.

$$\begin{array}{ll} \max_{x_i \geq 0} & u_i(x_i) \\ \text{s.t.} & p \cdot x_i \leq w_i \end{array}$$

First-Order Conditions for Pareto Optimality

The First Fundamental Welfare Theorem II

- Let α_i denote the Lagrange multiplier for each problem.
- The FOCs evaluated at the optimum (x^*, y^*) are, for all i, l ,

$$x_{li} : \quad \frac{\partial u_i}{\partial x_{li}} - \alpha_i p_l \begin{cases} \leq 0 & \text{if } x_{li} = 0 \\ = 0 & \text{if } x_{li} > 0 \end{cases}$$

- Note that letting $\mu_l = p_l$ for each l and $\delta_i = \frac{1}{\alpha_i}$ for each i , we obtain the same FOCs characterizing a PO allocation.
- Thus, (x^*, y^*) is PO iff it is a price equilibrium with transfers with respect to some positive price vector $p = (p_1, \dots, p_l)$.