SINGLE-OUTPUT TECHNOLOGY

<u>Prop.</u> Y is convex iff $f(x)$ is concave.

Prove this proposition

If Y is convex $\Rightarrow f(z)$ is concave.

$$Y = \{(-z, q) \in \mathbb{R}^m : q - f(z) \leq 0 \text{ and } z_1, \dots z_L \geq 0\}$$

take $z, z' \in \mathbb{R}^{m-1}_+$ and $\alpha \in [0,1]$ s.t.

$(-z, f(z)) \in Y$ and $(-z', f(z')) \in Y$

by convexity

$$\left(-[\alpha z + (1-\alpha) z'],\ \overbrace{\alpha f(z) + (1-\alpha) f(z')}^{\alpha q + (1-\alpha) q'}\right) \in Y$$

By def. of production set

$$\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$$

that implies $f(\cdot)$ concave.

If $f(\cdot)$ is concave $\Rightarrow Y$ is convex

Consider two prod. plans $(-z, q) \in Y$ and $(-z', q') \in Y$ by def.

$$q - f(z) \leq 0 \qquad q \leq f(z)$$

$$q' - f(z') \leq 0 \qquad q' \leq f(z')$$

Construct the convex combination:

$$\alpha q + (1-\alpha) q' \leq \alpha f(z) + (1-\alpha) f(z') \leq$$

$$\leq f(\alpha z + (1-\alpha) z')$$

by $f(\cdot)$ concave

Hence,

$$\alpha q + (1-\alpha) q' \leq f(\alpha z + (1-\alpha) z')$$

i.e. by def. of production set:

$$(-(\alpha z + (1-\alpha) z') ; \alpha q + (1-\alpha) q') \in Y$$

$$\alpha(-z, q) + (1-\alpha)(-z', q') \in \mathcal{Y}$$

i.e., $\mathcal{Y}$ is convex.

When $\mathcal{Y}$ is convex, PMP has a unique solution and FOCs are necessary and sufficient for $y(p)$.

PRODUCTION SINGLE OUTPUT

- Y exhibits constant returns to scale iff (is equivalent to) $f(\cdot)$ is homogeneous of degree one.

Prove this proposition.

If Y has CRS $\Rightarrow f(\cdot)$ is homog. of degree one

Let $z \in \mathbb{R}_+^{m-1}$ and $\alpha > 0$.

$$Y = \{(q, z) \in \mathbb{R}_+^m : q - f(z) \leq 0\}$$

the prod. plan $(-z, f(z)) \in Y$.

By constant returns to scale,

$$(-\alpha z, \alpha f(z)) \in Y$$

But $q - f(z) \leq 0$ equiv. $\alpha f(z) - f(\alpha z) \leq 0$

hence,

$$(1) \quad \alpha f(z) \leq f(\alpha z) \qquad \forall z, \forall \alpha$$

By CRS of Y, (let $z = \alpha z$ and $\alpha = \frac{1}{\alpha}$)

$$\frac{1}{\alpha} f(\alpha z) \leq f\left(\frac{1}{\cancel{\alpha}} \cdot \cancel{\alpha} z\right) = f(z)$$

$$(2) \quad f(\alpha z) \leq \alpha f(z)$$

Hence (1) & (2) $\Rightarrow$ $f(\alpha z) = \alpha f(z) \quad \forall \alpha \; \forall z$

If $f(\cdot)$ is homog. of degree 1 $\Rightarrow$ Y has CRS

Consider a prod. plan $(-z, q) \in Y$ and $\alpha > 0$.

By def. of Y: $q - f(z) \leq 0$ hence

$$q \leq f(z)$$

$$\Rightarrow \quad \alpha q \leq \alpha f(z) = f(\alpha z)$$

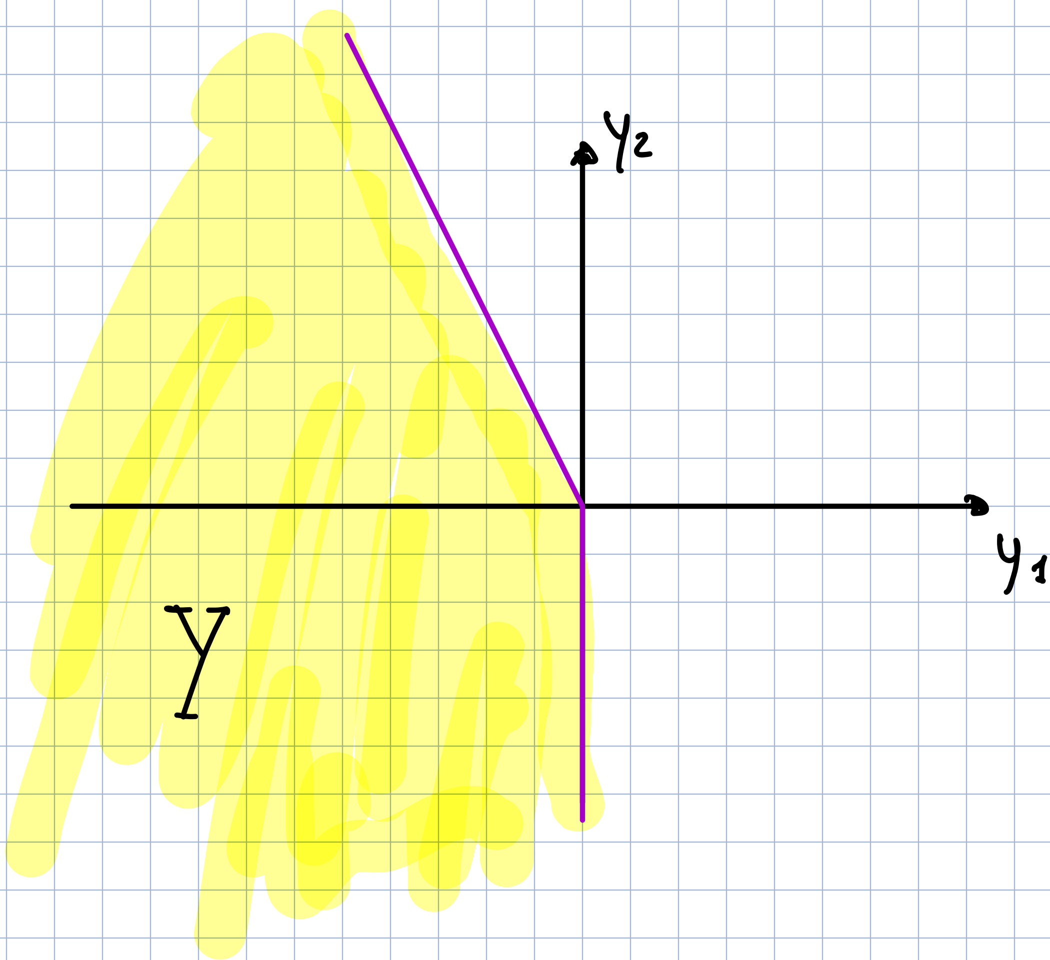
Hence,

$$(-\alpha z, \alpha q) \in Y \qquad \forall \alpha$$

this defines CRS.

PROFITS WITH CONSTANT or INCREASING RETURNS TO SCALE

$n = 2$



$f(y_1) = 2 y_1$

If $p_2 < p_1 \Rightarrow \nexists\ y_1 \neq 0$ that the firm is willing to demand

Hence, $\pi(p) = 0$

If $p_2 > p_1 \Rightarrow$ demand of input is unbounded $\Rightarrow \pi(p) = +\infty$