MATRIX NOTATION FOR SUBSTITUTION EFFECTS

$$p \cdot x(p,w) = \underset{(1\times 1)}{w}$$

$$\underset{L\times 1}{x(p,w)} = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_L(p,w) \end{bmatrix} \qquad \underset{L\times 1}{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix}$$

$$p \cdot x(p,w) = p_1 \cdot x_1(p,w) + p_2 \cdot x_2(p,w) + \dots + p_L x_L(p,w)$$

$$\underset{L\times L}{D_p x(p,w)} = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_L} \\ \vdots & & & \vdots \\ \frac{\partial x_L}{\partial p_1} & \frac{\partial x_L}{\partial p_2} & \cdots & \frac{\partial x_L}{\partial p_L} \end{bmatrix}$$

$$\underset{L\times 1}{D_w x(p,w)} = \begin{bmatrix} \frac{\partial x_1}{\partial w} \\ \vdots \\ \frac{\partial x_L}{\partial w} \end{bmatrix}$$

$$dw \qquad\qquad dx$$
$$1\times 1 \qquad\qquad L\times 1$$

with $L = 2$

$$dx = D_p\, x(p,w)\, dp + \underbrace{D_w\, x(p,w)\, dw}_{(L\times 1)\cdot(1\times 1)} \quad (1)$$

expands to

$$\begin{bmatrix} dx_1(p,w) \\ dx_2(p,w) \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} \begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial w} \end{bmatrix} dw$$

Differentiating Walras' law:

$$dw = dp \cdot x(p,w) = dp_1\, x_1(p,w) + dp_2\, x_2(p,w) =$$

$$= \underbrace{x(p,w)^T}_{(1\times L)} \cdot \underbrace{dp}_{(L\times 1)}$$

then, rewrite the second term in (1) as

$$\underset{L\times 1}{D_w\, x(p,w)\, dw} = \Big[\underbrace{\underset{L\times 1}{D_w\, x(p,w)} \cdot \underset{1\times L}{x(p,w)^T}}\Big] \cdot \underset{L\times 1}{dp}$$

where,

$$D_w\, x(p,w) \cdot x(p,w)^T = \begin{bmatrix} \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial w} \end{bmatrix} \cdot \begin{bmatrix} x_1(p,w) & x_2(p,w) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial w} x_1(p,w) & \frac{\partial x_1}{\partial w} \cdot x_2(p,w) \\ \frac{\partial x_2}{\partial w} x_1(p,w) & \frac{\partial x_2}{\partial w} x_2(p,w) \end{bmatrix}$$

$$\Big[D_p\, x(p,w) + \underset{L\times L}{D_w\, x(p,w) \cdot x(p,w)^T}\Big] =$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial w} \cdot x_1(p,w) & \frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial w} \cdot x_2(p,w) \\ \frac{\partial x_2}{\partial p_1} + \frac{\partial x_2}{\partial w} \cdot x_1(p,w) & \frac{\partial x_2}{\partial p_2} + \frac{\partial x_2}{\partial w} \cdot x_2(p,w) \end{bmatrix}$$

Rewrite equation (1) explicitely

$$dx = \left[D_p\, x(p,w) + D_w\, x(p,w) \cdot x(p,w)^T \right] dp \qquad (2)$$

the term in squared brackets is :

$$S(p,w) = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial w} \cdot x_1(p,w) & \frac{\partial x_1}{\partial p_2} + \frac{\partial x_1}{\partial w} \cdot x_2(p,w) \\ \frac{\partial x_2}{\partial p_1} + \frac{\partial x_2}{\partial w} \cdot x_1(p,w) & \frac{\partial x_2}{\partial p_2} + \frac{\partial x_2}{\partial w} \cdot x_2(p,w) \end{bmatrix}$$

the Slutsky substitution matrix - the differential version of WARP proposition implies that

$$dp \cdot dx \leq 0$$

hence, by (2)

$$\underset{(1\times L)}{dp} \cdot \underset{(L\times L)}{\left[D_p\, x(p,w) + D_w\, x(p,w)\, x(p,w)^T \right]} \underset{(L\times 1)}{dp} \leq 0$$

or

$$dp \cdot S(p,w)\, dp \leq 0$$

so $S(p,w)$ is negative semi-definite.