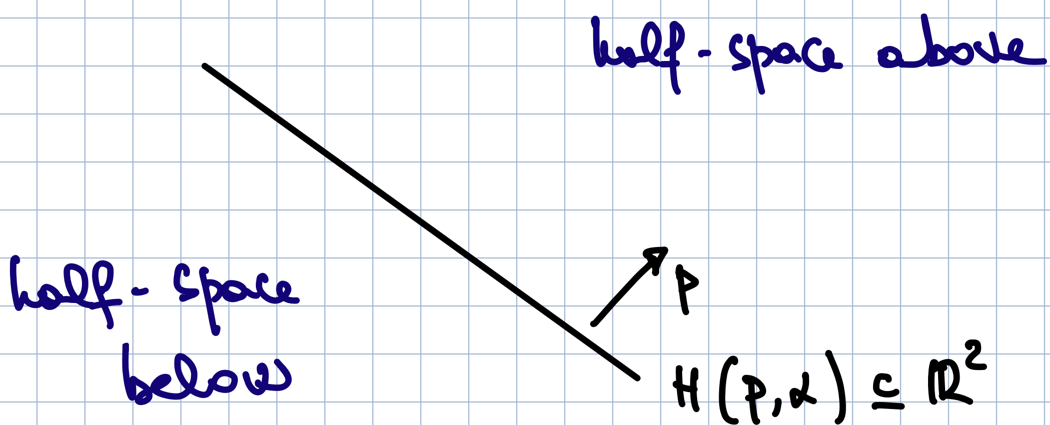
HYPERPLANE in $\mathbb{R}^L$

the set of solutions of one linear equation in L variables is called an hyperplane in $\mathbb{R}^L$, that is

$\forall\, p \in \mathbb{R}^L,\ p \neq 0$ and $\forall\, \alpha \in \mathbb{R}$ the set

$$H(p,\alpha) = \{ x \in \mathbb{R}^L : p_1 x_1 + p_2 x_2 + \dots + p_L x_L = \alpha \}$$

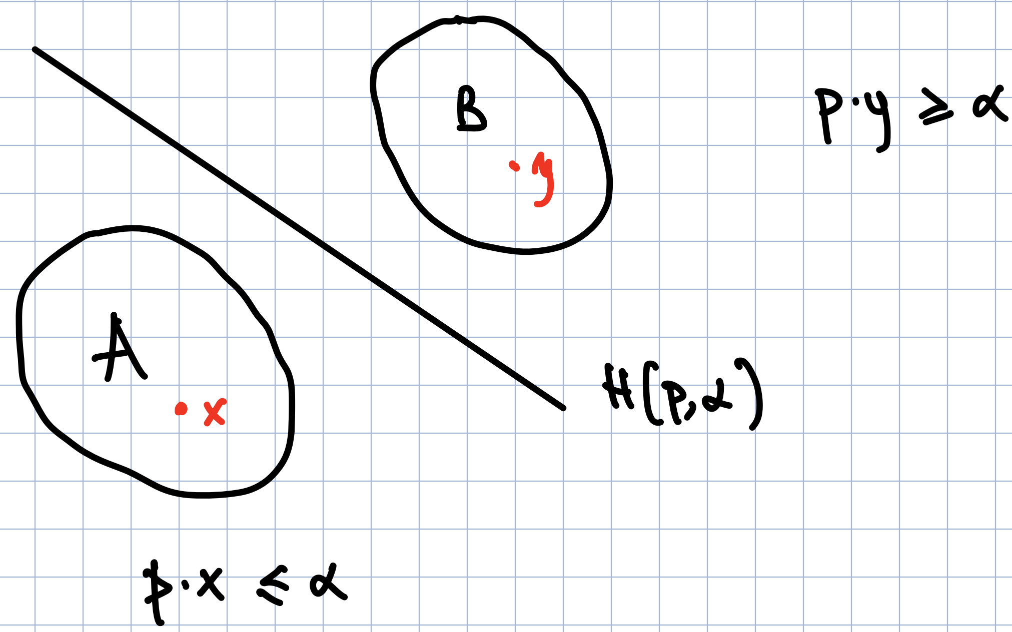
is a hyperplane. The vector p is called a normal to the hyperplane $H(p,\alpha)$



Pick $A, B \subseteq \mathbb{R}^2$ closed and convex.

We say that $H(p,\alpha) \subset \mathbb{R}^2$ separates the two sets A and B in $\mathbb{R}^L$ if

$$p \cdot x \leq \alpha \leq p \cdot y \qquad \forall\, x \in A \,, \ \forall\, y \in B$$



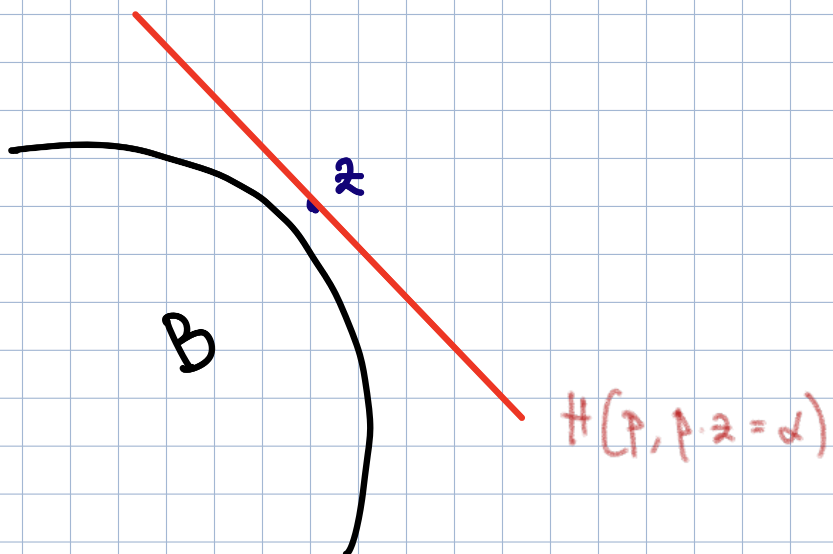
Supporting hyperplane theorem.

Prop. Let B be a convex subset of $\mathbb{R}^L$ and

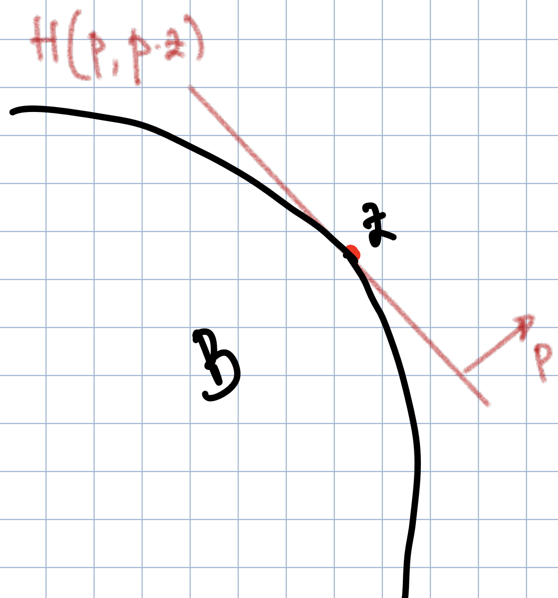
$$z \notin \operatorname{Int} B$$

then, there exists an hyperplane $H(p,\alpha)$, with $p \in \mathbb{R}^L$, $p \neq 0$ and $\alpha \in \mathbb{R}$, through z which separates z and B, i.e.

$$p \cdot z = \alpha \leq p \cdot y \qquad \forall\, y \in B$$



$z \notin B$



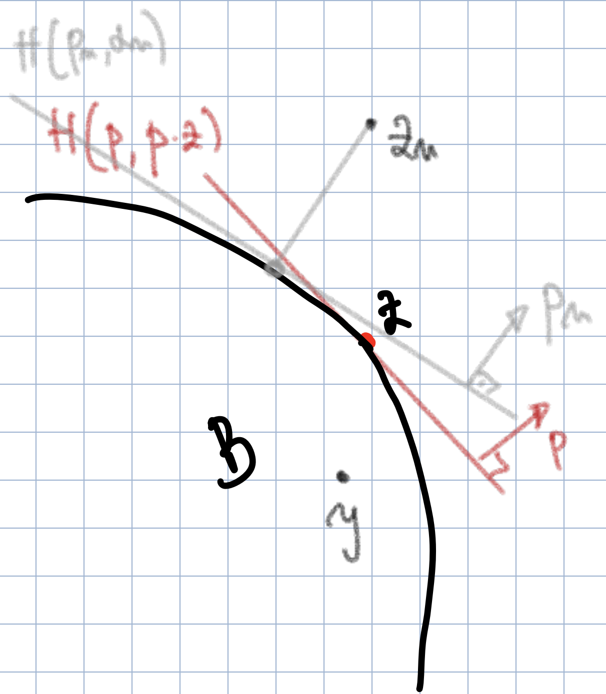
$z \in \bar{B}$

with

$\bar{B} \equiv$ closure of B

Proof. Let $z \notin \operatorname{Int} B$.
There exists a sequence (z_n) converging to z
with $z_n \notin \bar{B}$ $\forall n = 1, 2, \dots$.
Hence, there exist normals $p_n \in \mathbb{R}^L$, $p_n \neq 0$
and $\alpha_n \in \mathbb{R}$ s.t. $\forall n = 1, 2, \dots$

$$p_n \cdot z_n > \alpha_n \geq p_n \cdot y \quad \forall y \in B \qquad (1)$$



Let $\|p_n\| = 1$ $\forall n = 1, 2, \dots$
From $\{z_n\}$, (p^n, α_n) extract a converging sub-sequence $p^{n(k)} \xrightarrow[k]{} p$ $\quad p \neq 0$

$$\alpha_{n(k)} \xrightarrow[k]{} \alpha$$

so that taking limit $n \to +\infty$ of (1)
we get
$$p \cdot z \geq \alpha \geq p \cdot y \qquad \forall y \in B$$

since scalar product is a continuous function.

Hence, $H(p, \alpha = p \cdot z)$ is the desired hyperplane.