Ex 6.18 (Lindhal prices)

a) The cost share are such both consumers demand the same level of public good.

$$a_1 - b\tau_1 = a_2 - b\tau_2$$

but we use $\tau_2 + \tau_1 = 1$ to get:

$$G_1 = a_1 - b\tau_1 = a_2 - b\left(1 - \tau_1\right) = G_2$$

and solving the equation we obtain:

$$\tau_1 = \frac{a_1 - a_2 + b}{2b}$$

$$\tau_2 = \frac{a_2 - a_1 + b}{2b}$$

therefore by plugging τ_1 into G_1 we obtain:

$$G = \frac{a_1 + a_2 - b}{2}$$

b) The utility of consumer 1 is:

$$\begin{aligned} U_1 &= \log\left(x\right) + \log\left(G\right) \\ &= \log\left(M - \tau_1 G\right) + \log\left(G\right) \\ &= \log\left(M - \frac{a_1 - a_2 + b}{2b}\frac{a_1 + a_2 - b}{2}\right) + \log\left(\frac{a_1 + a_2 - b}{2}\right) \end{aligned}$$

and

$$\begin{aligned} U_2 &= \log\left(x\right) + \log\left(G\right) \\ &= \log\left(M - \frac{a_2 - a_1 + b}{2b}\frac{a_1 + a_2 - b}{2}\right) + \log\left(\frac{a_1 + a_2 - b}{2}\right) \end{aligned}$$

To find the Nash Equilibrium we need to find the best responses (BR) for both consumers with respect to their a_1 and a_2. The BR are give by the firs order condition

$$\frac{\partial U_1}{\partial a_1} = 0$$

that gives:

$$2a_1\left(a_1 + a_2 - b\right) = 4bM - \left(a_1 - a_2 + b\right)\left(a_1 + a_2 - b\right)$$

since consumer are symmetric (set the same a) in equilibrium we can use $a_1 = a_2 = a$ into the FOC and get

$$2a\,(2a-b)=4bM-b\,(2a-b)$$

that gives

$$a=\frac{1}{2}\sqrt{b^2+4bM}$$

then the equilibrium level of G is

$$G=\frac{1}{2}\left[\sqrt{b^2+4bM}-b\right]$$

EX 6.22 (VCG Mechanism)

a) the gross benefit are $v_1=-30$, $v_2=-10$, $v_3=50$. In the VCG mechanism the payoff of the player 1 is:

$$\pi_1=\begin{cases} v_1+r_2+r_3 & \text{if } r_1+r_2+r_3 \geq 0 \\ 0 & \text{if } r_1+r_2+r_3 < 0 \end{cases}$$

were r_2+r_3 are the side transfers to player 1. Now assume that 2 and 3 truthfully reveal their valuations. Then the payoff for player 1 becomes:

$$\pi_1=\begin{cases} v_1+v_2+v_3 & \text{if } r_1+40\geq 0 \\ 0 & \text{if } r_1+40<0 \end{cases}$$

It is possible to see that $r_1=v_1$ is weakly dominant for player 1, where r_1 does not directly enter the his payoff, it only affect indirectly his payoff by means of the necessary condition for the provision of the public good that is $r_1+40\geq 0$.

b) The provision of public is optimal because the total net benefit is positive, that is $v_1+v_2+v_3=10$ (or gross aggregate benefit is higher than its cost)

c) Assume that 1 and 2 collude by jointly setting $r_1=-27$ and $r_2=-8$. These report do not affect the equilibrium outcome such that the public good is delivered, in fact $-27-8+50\geq 0$. Moreover their payoffs are now:

$$\begin{aligned} \pi_1 &= -30-8+50=15 \\ \pi_2 &= -10-27+50=13 \end{aligned}$$

Hence if 3 truthfully reveals his valuation, 2 and 1 have the incentive to collude. In particular, their own misreport does not directly increase their own payoff, is the other's misreport that increases this payoff but each player need a collusive strategy to induce the other player to misreport and allow this increase in the payoff.

Ex 6.23 (Samuelson's rule)

We simply apply the Samuelson's rule that allows the efficient provision of the public good

$$MRS^1_{G,x}+MRS^2_{G,x}+MRS^3_{G,x}=10$$

that is:

$$MRS^1_{G,x} + MRS^2_{G,x} + MRS^3_{G,x} = \frac{\frac{\partial U^1}{\partial G}}{\frac{\partial U^1}{\partial x}} + \frac{\frac{\partial U^2}{\partial G}}{\frac{\partial U^2}{\partial x}} + \frac{\frac{\partial U^3}{\partial G}}{\frac{\partial U^3}{\partial x}} = 10 \tag{1}$$

where each MRS is $MRS^i_{G,x} = \frac{\frac{\partial U^i}{\partial G}}{\frac{\partial U^i}{\partial x}} = \frac{x_i}{G}$ then (1) becomes:

$$\frac{x_1}{G} + \frac{x_2}{G} + \frac{x_3}{G} = 10$$

so the SR becomes equal to

$$x_1 + x_2 + x_3 = 10G$$

now from the budget constraint of the economy that is:

$$x_1 + x_2 + x_3 + 10G = w_1 + w_2 + w_3$$

we get:

$$x_1 + x_2 + x_3 + 10G = 100$$

and by substituting into the SR and solving for G we get:

$$G = 5$$