Topic II b Optimal inter-temporal consumption and growth solved with optimal control

1) Y = F(K,AL)

F is an aggregate production function with constant returns to scale. Assume that capital doesn't depreciate so $\delta = 0$. Assume perfect competition so labour and capital are paid their marginal products. Thus $r_t = F_K(K_t, A_tL_t)$. Perfect competition and constant returns to scale imply that

2) $W_tL_t = Y_t - K_tr_t$

I assume that $\dot{L}_t = nL_t$ where n is the rate of growth of log employment and that $\dot{A}_t = gA_t$ so g is the rate of labour augmenting technological progress.

Assume that capital doesn't depreciate so $\delta = 0$ and

3) $\dot{K}_t = F(K_t, A_tL_t) - C_t = F(K_t, A_tL_t) - C_{pt}L_t$

where C_t is total consumption and C_{pt} is consumption per capita. To consider consumption savings choices, we really have to divide workers into households since consumption is a really done by a household. (Households are groups of people who live together and share living expenses). Assume that households last forever and we can even assume that the number of households is a constant H. The number of worker/consumers per household L/H grows at rate n. C_p is equal to household consumption per person so it is total consumption divided by L (this is C according to Romer). Within the household all altruistically agree to maximize

4) $V_t = \int_0^{\infty} e^{-\rho(t)} u(C_{pt})(L_t/H)dt =$

$$(L_O/H) \quad V_t = \int_0^{\infty} e^{(n-\rho)(t)} u(C_{pt})dt$$

Subject to the infinitely many constraints given by equation 5 for each t.

5) $\dot{K}_t/H = F(K_t/H, A_t L_t/H) - C_t/H = (F(K_t, A_t L_t) - C_{pt} L_t)/H$

Equation 5 is just equation 3 divided by the irritating H so K/H is the wealth of the household.

For each t define a Lagrange multiplier Q_t so constrained maximization is equivalent to unconstrained maximization of

$$\int_0^\infty e^{-\rho(t)} u(C_{pt})(L_t/H)-Q_t[\ \dot{K}_t/H-\left(F(K_t,A_tL_t)+C_{pt}L_t\right)/H\ ]\ dt$$

Just to simplify a little, maximizing H times something is the same as maximizing that something, so the problem is unconstrained maximization of 6 (where I also multiplied through by $-Q_t$

6)
$$\int_0^\infty e^{-\rho(t)} u(C_{pt})L_t- Q_t\dot{K}_t+Q_t\left(F(K_t,A_tL_t)\right)-Q_tC_{pt}L_t¿\ ]\ dt$$

Let's integrate by parts.

$$d\frac{Q_tK_t}{dt}=\dot{Q}_tK_t+Q_t\dot{K}_t$$

So $Q_TK_t-Q_0K_0=\int_0^T(\dot{Q}_tK_t+Q_t\dot{K}_t)dt$

and taking the limit as T goes to infintity and rearanging a little gives

7) $-\lim_{t\to\infty}Q_tK_t+Q_0K_0-\int_0^\infty\dot{Q}_tK_tdt=-\int_0^\infty Q_t\dot{K}_t$

This makes it possible to get rid of the K dot term in the maximand 6 and get to 0

8) $-\lim_{t\to\infty} Q_t K_t + Q_0 K_0$ +

$\int_0^\infty$ $e^{-\rho(t)}$ $u(C_{pt})L_t$ + $\dot{Q}_t K_t + Q_t(F(K_t, A_t L_t)) - Q_t C_{pt} L_t$ dt

Now define

$q_t = e^{\rho t} Q_t$

So $Q_t = e^{-\rho t} q_t$ and

9) $\dot{Q}_t = e^{-\rho t}(-\rho q_t + \dot{q}_t)$

This gives equation 10

10) $-\lim_{t\to\infty} Q_t K_t + Q_0 K_0$ +

$\int_0^\infty$ $e^{-\rho(t)}$ $[u(C_{pt})L_t - \rho q_t K_t + \dot{q}_t K_t + q_t(F(K_t, A_t L_t)) - q_t C_{pt} L_t]$ dt

If and **only if** we can ignore the top terms $-\lim_{t\to\infty} Q_t K_t + Q_0 K_0$ because they give a constant which doesn't depend on C_{pt} this gives a lot of first order conditions for a maximum, for each t there is a first order condition with respect to C_{pt}

11) $u'(C_{pt}) - q_t = 0$

And a first order condition with respect to K_t

12) $0 = -\rho q_t + \dot{q}_t + q_t r_t$

equation 12 implies

13) $\frac{\dot{q}_t}{q_t} = \rho - r_t$

11 and 13 combine to give the Euler equation

14 $\frac{\dot{u}'(\dot{C}_{pt})}{u'(C_{pt})} = \rho - r_t$

But wait I assumed that $-\lim_{t\to\infty} Q_t K_t + Q_0 K_0$ is a constant which I can ignore when taking derivatives. For a proposed solution to be the correct solution, the Euler equation is not sufficient. It is also necessary that that messy term is a constant. Q_0K_o really is given, so it's not a problem, but I need that $-\lim_{t\to\infty} Q_t K_t$ is a constant.

It is a constant if and only if it is zero. That means a necessary condition for a solution is that the lagrange multiplier times how much the household wants more capital a ttime t goes to zero as t goes to infinity.
This is called the transversality condition. That's a long word, but it is actually equivalent to the assumption that the intertemporal budget constraint holds with equality.
It is possible to come up with an incorrect solution in which the Euler equation always holds, but utility V is not maximized, because the household has a non binding budget constraint and could just consume more.

An advantage of using the intertemporal budget constraint and one big lagrance multiplier (not one for each t times the instantaneous budget constraint) is that this makes it clearer why the Euler equation is necessary but not sufficient for a solution.