

Mixed-Strategies

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Matching Pennies Revisited

		Player 2	
		H	T
Player 1	H	(<u>1</u> , -1)	(-1, <u>1</u>)
	T	(-1, <u>1</u>)	(<u>1</u> , -1)

- The concept of *NE* (in pure strategies) fails in these games
- One player would like to **outguess** the other. Examples are
 - Poker → How often to bluff?
 - Battle → Attack by land or sea?
- No *pure-strategy NE* exists in these games, as there is **uncertainty** concerning what the other players will do.
- Uncertainty is captured by the concept of **mixed strategy**.

Mixed Strategies

Definition (*mixed strategy*):

In the normal-form game $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$ suppose

$S_i = \{s_{i1}, \dots, s_{iK}\}$ (s_{ik} is a generic pure strategy k for player i).

Then a **mixed strategy** for player i is a **probability distribution**

$p_i = (p_{i1}, \dots, p_{iK})$, where $0 \leq p_{ik} \leq 1$ for $k = 1, \dots, K$ and

$p_{i1} + \dots + p_{iK} = 1$.

- We denote the game accounting for mixed strategies as *Gamma*:

$$\Gamma = \{N, P_i(S_i)_{i \in N}, \nu_{i \in N}\}$$

- If the probability attached to s_{ik} is $p_{ik} = 1$, then the player is playing a pure strategy.

IESDS Revisited (1)

Consider the following generic game:

		Player 2	
		$L(y)$	$R(1-y)$
Player 1	T	$(3, -)$	$(0, -)$
	M	$(0, -)$	$(3, -)$
	B	$(1, -)$	$(1, -)$

- There is **no dominant** pure strategy. However, suppose that Player 1 thinks that 2 adopts the **mixed strategy** y .
- We compute Player 1's **expected payoffs** from T , M , and B :
 - $\nu(T, y) = 3y + 0(1 - y) = 3y$.
 - $\nu(M, y) = 0y + 3(1 - y) = 3 - 3y$.
 - $\nu(B, y) = y + (1 - y) = 1$.

IESDS Revisited (2)

- Suppose that Player 1 plays T and M with probability $1/2$, then she gets:

$$\frac{3}{2}y + \frac{3}{2} - \frac{3}{2}y = \frac{3}{2} > 1 = v(B, y).$$

- So B is strictly dominated by the mixed strategy $\left(\frac{1}{2}T + \frac{1}{2}M\right)$ for any y such that $0 \leq y \leq 1$.

Mixed-Strategy NE (1)

- Let $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{ik}, \dots, \sigma_{iK})$ the set of all mixed strategies available to player i .
- Let $\sigma_{ik} = p_{ik}s_{ik}$ be a generic mixed strategy for player i with $\sum_{k=1}^K p_{ik} = 1, p_{ik} \geq 0$.

Definition (*NE in mixed strategies*):

In the normal-form game $\Gamma = \{N, P_i(S_i)_{i \in N}, \nu_{i \in N}\}$ a mixed strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_N^*)$ constitutes a *NE* if for every $i = 1, \dots, N$

$$\nu_i(\sigma_i^*, \sigma_{-i}^*) \geq \nu_i(\sigma_i', \sigma_{-i}^*)$$

for every $\sigma_i' \in P_i(S_i)$.

Mixed-Strategy NE (2)

- Mixed-strategy *NE* represent uncertainty by one player with respect to what the other player will do.
- This is why they choose to **randomise** over their pure strategies.
- Simply an extension of pure-strategy *NE*, as it requires that a mixed strategy is a **best-response** to the opponent's equilibrium mixed strategy.

Caution: there exists mixed-strategy equilibria where best responses are played with probability 1, i.e. pure-strategy NE.

Mixed-Strategy Equilibrium in Matching Pennies (1)

- Let $0 \leq x \leq 1$ and $0 \leq y \leq 1$ be the probabilities that player 1 and 2 choose *Head*.

		Player 2	
		y	1-y
		H	T
Player 1 x	H	(1, -1)	(-1, 1)
	1-x	T (-1, 1)	(1, -1)

- We divide this problem in 6 cases:
 - $x = 1$, player 1 plays *H* with certainty \rightarrow No equilibrium
 - $x = 0$, player 1 plays *T* with certainty \rightarrow No equilibrium
 - $x \in (0, 1) \rightarrow$ Player 1 randomises
 - $y = 1$, player 2 plays *H* with certainty \rightarrow No equilibrium
 - $y = 0$, player 2 plays *T* with certainty \rightarrow No equilibrium
 - $y \in (0, 1) \rightarrow$ Player 2 randomises

Mixed-Strategy Equilibrium in Matching Pennies (2)

- In order for a *mixed-strategy NE* to arise, we need:
 1. Player 1 has the same expected payoff (payoff times probability) from playing H or T when player 2 plays the mixed strategy y , i.e.

$$\nu_1(H, y) = \nu_1(T, y).$$

2. Player 2 has the same expected payoff (payoff times probability) from playing H or T when player 1 plays the mixed strategy x , i.e.

$$\nu_2(x, H) = \nu_2(x, T).$$

- If these conditions are not satisfied, players will prefer some pure strategy, as shown below.

Mixed-Strategy Equilibrium in Matching Pennies (2)

- Take the expected payoffs of player 1:

$$\nu_1(H, y) = 1 \times y + (-1) \times (1 - y) = 2y - 1.$$

$$\nu_1(T, y) = (-1) \times y + 1 \times (1 - y) = 1 - 2y.$$

- Let us suppose $\nu_1(H, y) > \nu_1(T, y)$. Then

$$2y - 1 > 1 - 2y \implies 4y > 2 \implies y > \frac{1}{2}.$$

If $y > 1/2$ player 1 chooses H with certainty.

- Let us suppose instead $\nu_1(H, y) < \nu_1(T, y)$. Then:

$$2y - 1 < 1 - 2y \implies 4y < 2 \implies y < \frac{1}{2}.$$

If $y < 1/2$ player 1 chooses T with certainty.

Mixed-Strategy Equilibrium in Matching Pennies (3)

- Take the expected payoffs of player 2:

$$\nu_2(x, H) = (-1) \times x + 1 \times (1 - x) = 1 - 2x.$$

$$\nu_2(x, T) = 1 \times x + (-1) \times (1 - x) = 2x - 1.$$

- Let us suppose $\nu_2(x, H) > \nu_2(x, T)$. Then

$$1 - 2x > 2x - 1 \implies 4x < 2 \implies x < \frac{1}{2}.$$

If $x < 1/2$ player 2 chooses H with certainty

- Let us suppose instead $\nu_2(x, H) < \nu_2(x, T)$. Then:

$$1 - 2x < 2x - 1 \implies 4x > 2 \implies x > \frac{1}{2}.$$

If $x > 1/2$ player 1 chooses T with certainty.

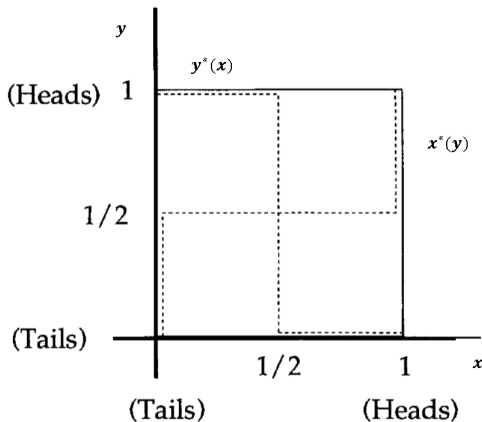
Mixed-Strategy Equilibrium in Matching Pennies (3)

- The only probabilities such that Players 1 and 2 are willing to randomise over their pure strategies are respectively $y = 1/2$ and $x = 1/2$.
- There is only 1 *NE* in this game, it is in mixed strategies, and can be denoted as

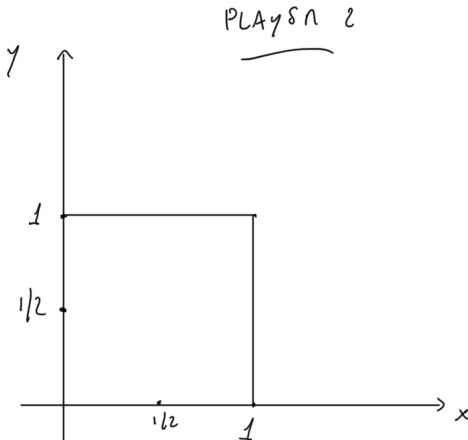
$$\left(\frac{1}{2}, \frac{1}{2}\right) \text{ or } \left(\frac{1}{2}H + \frac{1}{2}T; \frac{1}{2}H + \frac{1}{2}T\right).$$

Mixed-Strategy Equilibrium in Matching Pennies (4)

- If $x < 1/2$, Player 2's best response is H .
- If $x > 1/2$, Player 2's best response is T .
- If $x = 1/2$, Player 2 is indifferent and randomises.
- If $y < 1/2$, Player 1's best response is T .
- If $y > 1/2$, Player 1's best response is H .
- If $y = 1/2$, Player 1 is indifferent and randomises.

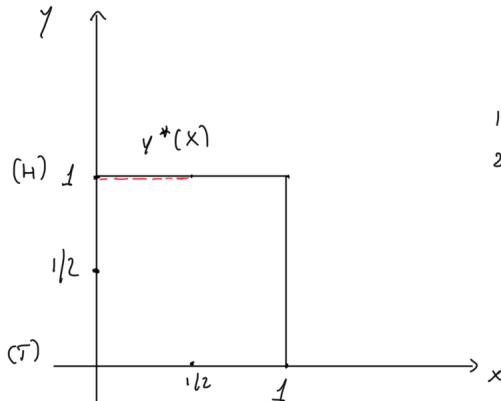


PROBABILITY
THAT 2
PLAYS H



PROBABILITY
THAT 1 PLAYS H

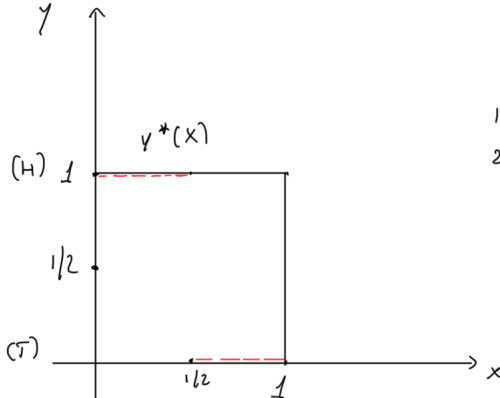
PROBABILITY
THAT 2
PLAYS H



1/2 $x < 1/2$
2's BEST RESPONSE
IS H

PROBABILITY
THAT 1 PLAYS H

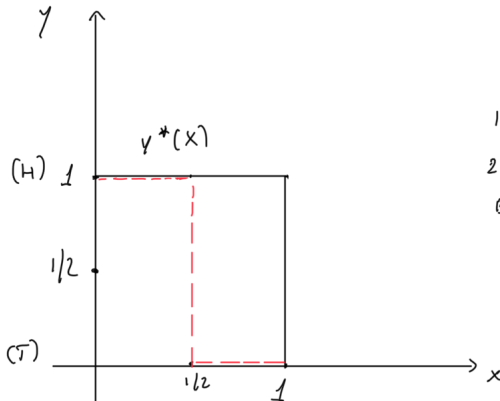
PROBABILITY
THAT 2
PLAYS H



1/2 $x > 1/2$
2's BEST RESPONSE
IS T

PROBABILITY
THAT 1 PLAYS H

PROBABILITY
THAT 2
PLAYS H

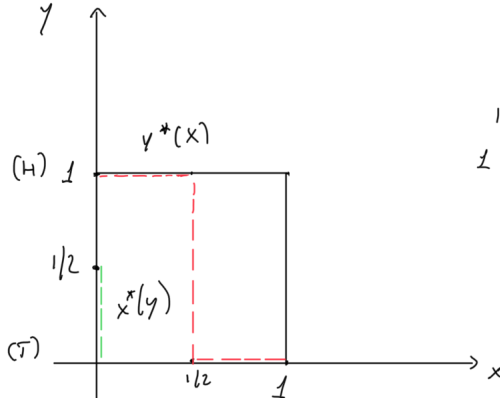


if $x \geq 1/2$,
2 IS INDIFFERENT
BETWEEN H AND T

PROBABILITY
THAT 1 PLAYS H

PLA ysm 1

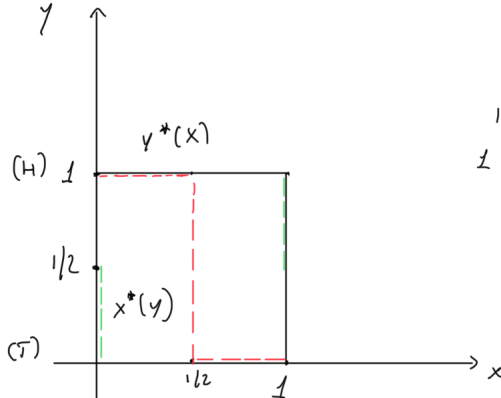
PROBABILITY
THAT 2
PLAYS H



1/2 $y < 1/2$
1's BEST RESPONSE
IS T

PROBABILITY
THAT 1 PLAYS H

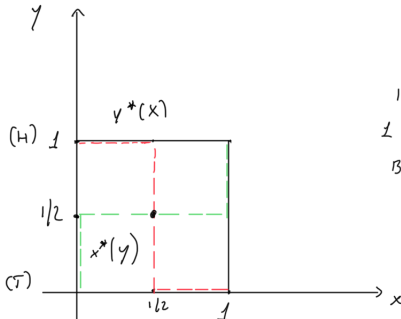
PROBABILITY
THAT 2
PLAYS H



if $y > 1/2$
1's best response
is H

PROBABILITY
THAT 1 PLAYS H

PROBABILITY
THAT 2
PLAYS H



if $y = 1/2$
1 IS INDIFFERENT
BETWEEN H AND T

PROBABILITY
THAT 1 PLAYS H

THE ONLY x, y SUCH THAT BOTH PLAYERS
ARE WILLING TO RANDOMISE OVER THEIR
TWO STRATEGIES ARE

$$x = y = 1/2$$

Mixed-Strategy Equilibrium in the Battle of Sexes (1)

		Pat	
		y	1-y
		Opera	Fight
Carl x	Opera	(2, 1)	(0, 0)
	1-x Fight	(0, 0)	(1, 2)

- In this case we have two *pure-strategy NE*. They occur when

$$x = 1, y = 1 \text{ and } x = 0, y = 0.$$

- Are there any *mixed-strategy NE*?

Mixed-Strategy Equilibrium in the Battle of Sexes (2)

- Take the expected payoffs for Carl:

$$\nu_C(O, y) = 2 \times y + (0) \times (1 - y) = 2y,$$

$$\nu_C(T, y) = 0 \times y + 1 \times (1 - y) = 1 - y.$$

$$2y = 1 - y \implies y = 1/3.$$

- Take the expected payoffs for Pat:

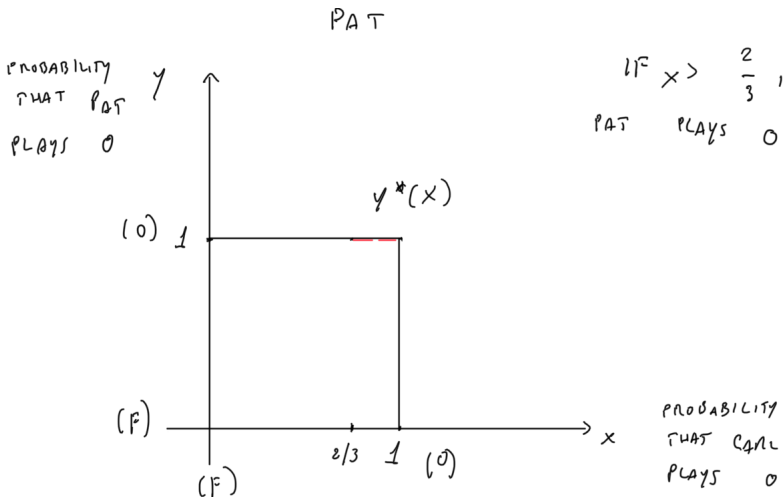
$$\nu_C(x, O) = 1 \times x + (0) \times (1 - x) = x,$$

$$\nu_C(x, T) = 0 \times x + 2 \times (1 - x) = 2 - 2x.$$

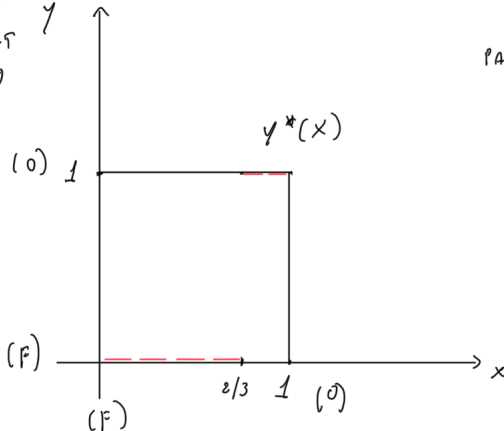
$$x = 2 - 2x \implies x = 2/3.$$

- The *mixed-strategy NE* is

$$\left(\frac{2}{3}, \frac{1}{3}\right) \text{ or } \left(\frac{2}{3}O + \frac{1}{3}F; \frac{1}{3}O + \frac{2}{3}F\right).$$



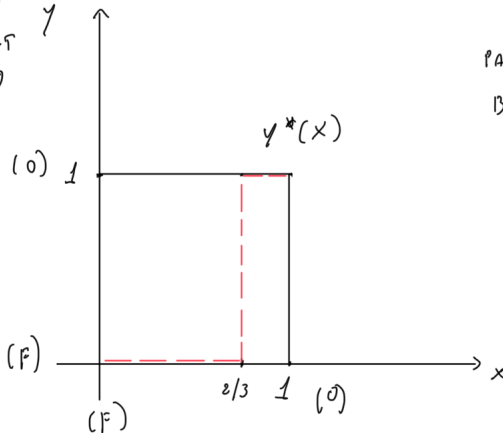
PROBABILITY
THAT PAT
PLAYS 0



IF $x < \frac{2}{3}$,
PAT PLAYS F

PROBABILITY
THAT CARL
PLAYS 0

PROBABILITY
THAT PAT
PLAYS O

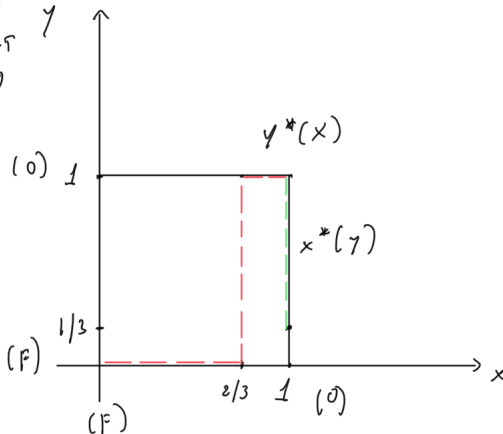


CANL

IF $x = \frac{2}{3}$,
PAT IS INDIFFERENT
BETWEEN O AND F

PROBABILITY
THAT CANL
PLAYS O

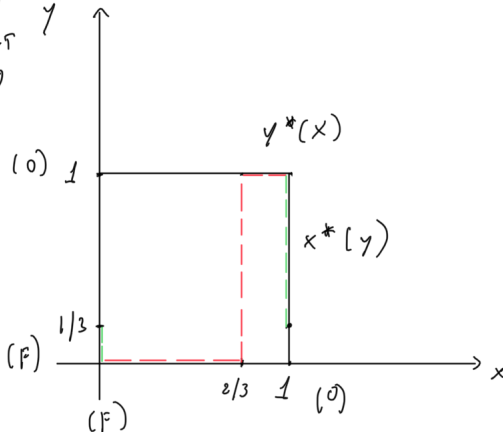
PROBABILITY
THAT P1
PLAYS 0



if $y > \frac{1}{3}$,
CAN PLAY
UPON

PROBABILITY
THAT CAN
PLAY 0

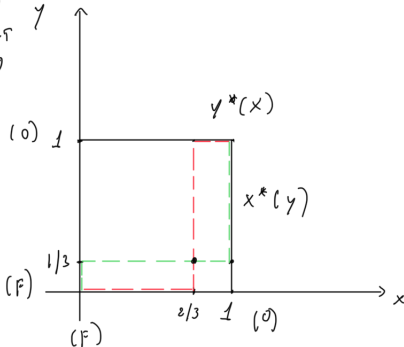
PROBABILITY
THAT PLAY
PLAYS 0



IF $y < \frac{1}{3}$,
CAN PLAY
FIGHT

PROBABILITY
THAT CAN
PLAYS 0

PROBABILITY
THAT CAT
PLAYS 0



IF $y = \frac{1}{3}$,
CAT IS INDIFFERENT
BETWEEN OPEN
AND FIGHT

PROBABILITY
THAT CAT
PLAYS 0

We have 3 N.E., two of which
are in pure strategies, and one in
mixed strategies

$$\left\{ \frac{2}{3} 0 + \frac{1}{3} F; \frac{1}{3} 0 + \frac{2}{3} F \right\}$$

NE in Prisoners' Dilemma (1)

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1, -1)$	$(-9, \underline{0})$
	Confess	$(\underline{0}, -9)$	$(\underline{-6}, \underline{-6})$

Prisoner 1's best response to:

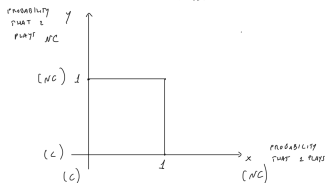
- C is C ($-6 > -9$).
- NC is C ($0 > -1$).

Player 2's best response to:

- C is C ($-6 > -9$).
- NC is C ($0 > -1$).

Remember that C is **dominant** for both players.

Prisoner 2



		$\frac{2}{3} \quad y$	$\frac{1}{3} \quad 1-y$
		NC	C
X	NC	$(-1, -1)$	$(-9, 0)$
1	C	$(0, -5)$	$(-6, -6)$
	$1-x$		

$$U_1(NC, y) = -y - 3(1-y) = 3y - y - 3 = 2y - 3$$

$$U_1(C, y) = 0 - 6(1-y) = 6y - 6$$

$$U_2(x, NC) = -x - 9(1-x) = 8x - 9$$

$$U_2(x, C) = 0 - 6(1-x) = 6x - 6$$

Let us check when the two expected payoffs coincide

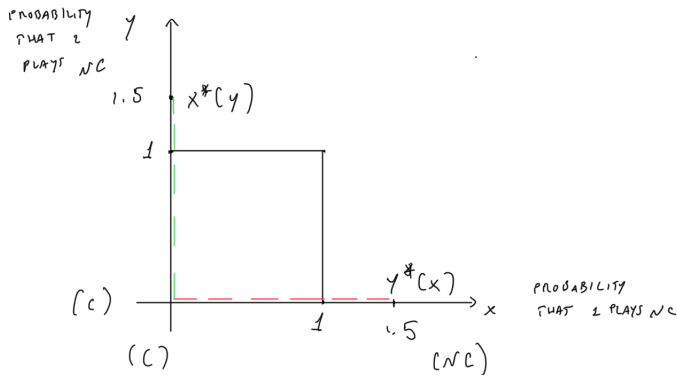
$$U_1(NC, y) = U_1(C, y)$$

$$2y - 3 = 6y - 6$$

$$2y = 3 \quad \Rightarrow \quad y = \frac{3}{2} \notin [0, 1]$$

$$\text{similarly} \quad x = \frac{3}{2} \notin [0, 1]$$

Prisoners 2



There is only one N.E. in
 pure strategies, i.e.
 $x, y = 0$ (c, c)

Existence of Nash Equilibrium

Theorem (Nash, 1950): *In the n -player normal-form game $G = \{S_1, \dots, S_n, u_1, \dots, u_n\}$, if n is finite and S_i is finite for every i then there exists at least one Nash equilibrium, possibly involving mixed strategies.*