Static Games of Complete Information

Lorenzo Ferrari

University of Rome Tor Vergata

General Information

- **Schedule**

  Mo (15:00 - 17:00), We (15:00 - 17:00), Fr (15:00 - 17:00).

- **Office Hours**: by appointment.

- **References**:

  Gibbons, R. (1992), Game theory for applied economists, Princeton University Press.

- **Exam**:
  - 70% written exam, 30% Social Capital project (deadline in December);
  - Grade is averaged (50%) with the one in **Industrial Organisation**.

- **Problem Sets**: uploaded at the end of every week, solved on Monday in class.

Outline

- Introduction
- Normal-Form
- Examples
- IESDS
- NE
- Cournot
- Bertrand
- Tragedy of the Commons
- Mixed Strategies

- Game theory is the study of **multiperson decision problem**:

  1. Involves two or more players
  2. Each player's payoffs (utility) depends on the decision of the others.

- Many applications in fields of economics:
  1. Industrial Organisation
  2. Labour and Financial Economics
  3. International Economics

- We study **four categories** of games:
  1. Static Games of Complete Info $\rightarrow$ Nash Equilibrium
  2. Dynamic Games of Complete Info $\rightarrow$ Subgame-perfect Nash Equilibrium
  3. Static Games of Incomplete Info $\rightarrow$ Bayesian Nash Equilibrium
  4. Dynamic Games of Incomplete Info (if we have time) $\rightarrow$ Perfect Bayesian Nash Equilibrium

Assumptions

- **Static**: players choose *simultaneously* what to play (alternatively, each of them does not know the others' choice)
- **Complete Info**: each player's *payoff function* (depending on all players' decisions) is *common knowledge*

Outline

- Introduction
- Normal-Form
- Examples
- IESDS
- NE
- Cournot
- Bertrand
- Tragedy of the Commons
- Mixed Strategies

- It is convenient to describe simultaneous games in *normal form*. This requires to specify the:

  1. **Players** of the game
  $$N = \{1, 2, ..., i-1, i, i+1, ..., n\}.$$
  2. **Strategies** available to each player (in her set of strategies)
  $$s_i \in S_i.$$
  3. **Payoffs** received by each player for each combination of strategies available to players
  $$u_i = u_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) = u_i(s_{-i}, s_i).$$

Definition: *The normal-form representation of an n-player game specifies the players, the players' strategy spaces* $S_1, ..., S_n$ *and their payoff functions* $u_1, ..., u_n$. *We denote this game* $G = \{1, 2, ..., N, S_1, ..., S_n, u_1, ..., u_n\}$

Outline

Matching Pennies

- Two players simultaneously choose **Head** or **Tail** on a penny
- If pennies match, player 2 pays 1. Otherwise, player 1 plays 2

		Player 2	
		H	T
Player 1	H	$(1, -1)$	$(-1, 1)$
	T	$(-1, 1)$	$(1, -1)$

- $N = \{Player1, Player2\}$
- $S_1 = S_2 = \{H, T\}$
- $u_1(H, H) = 1, u_1(H, T) = -1, u_1(T, H) = -1, u_1(T, T) = 1$
- $u_2(H, H) = -1, u_2(H, T) = 1, u_2(T, H) = 1, u_2(T, T) = -1$

Meeting in Rome

- Marco and Giulia are supposed to meet in Rome, but they do not remember whether in *Termini station* or *Tiburtina station*. They cannot communicate
- If they meet, they get \$100 each. Otherwise, they get nothing

		Giulia	
		Termini	Tiburtina
Marco	Termini	$(100, 100)$	$(0, 0)$
	Tiburtina	$(0, 0)$	$(100, 100)$

- $N = \{Marco, Giulia\}$
- $S_1 = S_2 = \{Te, Ti\}$
- $u_M(Te, Te) = 100, u_M(Te, Ti) = 0, u_M(Ti, Te) = 0, u_M(Ti, Ti) = 100$
- $u_G(Te, Te) = 100, u_G(Te, Ti) = 0, u_G(Ti, Te) = 0, u_G(Ti, Ti) = 100$

The Prisoners' Dilemma

- Two suspects are arrested and charged with a crime

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1,-1)$	$(-9,0)$
	Confess	$(0,-9)$	$(-6,-6)$

- $N = \{Prisoner1, Prisoner2\}$
- $S_1 = S_2 = \{C, NC\}$
- $u_1(C,C) = -6, u_1(C,NC) = 0, u_1(NC,C) = -9, u_1(NC,NC) = -1$
- $u_2(C,C) = -6, u_2(C,NC) = -9, u_2(NC,C) = 0, u_2(NC,NC) = -1$

The Battle of Sexes

- Carl and Pat must choose where to spend the evening. They love being together

Pat

Carl	Opera	Fight
Opera	$(2, 1)$	$(0, 0)$
Fight	$(0, 0)$	$(1, 2)$

- $N = \{Carl, Pat\}$
- $S_1 = S_2 = \{Opera, Fight\}$
- $u_C(O, O) = 2, u_C(O, F) = 0, u_C(F, O) = 0, u_C(F, F) = 1$
- $u_P(O, O) = 1, u_P(O, F) = 0, u_P(F, O) = 0, u_P(F, F) = 2$

Outline

Iterated Elimination of Strictly Dominated Strategies (1)

Player 2

		Left	*Middle*	*Right*
Player 1	*Up*	$(1,0)$	$(1,2)$	$(0,1)$
	Down	$(0,3)$	$(0,1)$	$(2,0)$

Player 2

		Left	*Middle*	*Right*
Player 1	*Up*	$(1,0)$	$(1,2)$	$(0,1)$
	Down	$(0,3)$	$(0,1)$	$(2,0)$

- Middle gives player 2 a *higher payoff* **irrespective** of what Player 1 chooses ($2>1$ and $1>0$)
- Since Player 2 is rational, he will **never play** *Right*. The latter is **strictly dominated** by *Middle*

Iterated Elimination of Strictly Dominated Strategies (2)

Player 2

Player 1		*Left*	*Middle*
	Up	$(1,0)$	$(1,2)$
	Down	$(0,3)$	$(0,1)$

Player 2

Player 1		*Left*	*Middle*
	Up	$(1,0)$	$(1,2)$

- Player 1 knows that Player 2 will never play *right*
- *Up* gives player 1 a *higher payoff* **irrespective** of what Player 2 chooses $(1 > 0$ and $1 > 0)$
- Since Player 1 is rational, he will **never play** *Down*. The latter is **strictly dominated** by *Up*

Iterated Elimination of Strictly Dominated Strategies (3)

Definition:

In the normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$ *let* $s_i^{'}$ *and* $s_i^{''}$ *feasible strategies for player i* ($s_i^{'}, s_i^{''} \in S_i$). *Strategy* $s_i^{'}$ *is* **strictly dominated** *by strategy* $s_i^{''}$ *if* **for each feasible combination** *of the other players' strategies, i's payoff from playing* $s_i^{'}$ *is* **strictly less** *than i's payoff from playing* $s_i^{''}$*:*

$$u_i(s_1, ..., s_{i-1}, s_i^{'}, s_{i+1}, ..., s_n) < u_i(s_1, ..., s_{i-1}, s_i^{''}, s_{i+1}, ..., s_n)$$

for each $(s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ that can be constructed from the other players' strategy spaces $S_1, ..., S_{i-1}, S_{i+1}, ..., S_n$ $(\forall s_j \in S_j, j \neq i)$.

Drawbacks (1)

- *IESDS* requires to assume that **it is common knowledge** that players are rational (not only all players are rational, but all players know that all players are rational, and all players know that all players know that all players are rational, *ad infinitum*)
- It is not always a **good solution concept**, as it can lead to very **imprecise predictions**
- This is not desirable as our task is to predict the outcome of the game as precisely as possible

Drawbacks (2)

		Player 2		
		Left	*Center*	*Right*
Player 1	*Top*	$(0,4)$	$(4,0)$	$(5,3)$
	Middle	$(4,0)$	$(0,4)$	$(5,3)$
	Bottom	$(3,5)$	$(3,5)$	$(6,6)$

- There is **no strictly dominated strategy**!
- We resort to a stronger concept, the one of **Nash Equilibrium**

Outline

Nash Equilibrium

Definition (*pure-strategy NE*):

In the n-player normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$ *the strategies* $(s_1^*, ..., s_n^*)$ *are a Nash Equilibrium if, for each player* i*,* s_i^* *is (at least tied for) player* i*'s best response to the strategies specified for the* $n-1$ *other players,* $(s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*)$*:*

$$u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \geq u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

for every $s_i \in S_i$. In other words, s_i^* solves

$$\max_{s_i \in S_i} u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

Notice: Given the other players' strategies!

Conditions for Nash Equilibrium

Two conditions characterise the concept of *NE*:

1. Each player must be playing the **best response** to the other players' strategies
2. There is **no profitable deviation** from the strategies chosen in equilibrium. If $s_1^{'}, ..., s_n^{'}$ is not a *NE*, there exists a strategy $s_i^{''}$ such that

$$u_i(s_1^{'}, ..., s_{i-1}^{'}, s_i^{'}, s_{i+1}^{'}, ..., s_n^{'}) \leq u_i(s_1^{'}, ..., s_{i-1}^{'}, s_i^{''}, s_{i+1}^{'}, ..., s_n^{'})$$

The prediction achieved using this concept is **strategically stable** (also said **self-enforcing**)

NE in our Previous Example

Player 2

Player 1		*Left*	*Center*	*Right*
	Top	$(0, \underline{4})$	$(\underline{4}, 0)$	$(5, 3)$
	Middle	$(\underline{4}, 0)$	$(0, \underline{4})$	$(5, 3)$
	Bottom	$(3, 5)$	$(3, 5)$	$(\underline{6}, \underline{6})$

The strategies (*Bottom*, *Right*) represent the only *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy
- There is no strict incentive to *unilaterally deviate* from the equilibrium

NE in Prisoners' Dilemma

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1,-1)$	$(-9,\underline{0})$
	Confess	$(\underline{0},-9)$	$(\underline{-6},\underline{-6})$

- When Player 2 chooses NC, the best response for Player 1 (green) is C. When Player 2 chooses C, the best response for Player 1 is C
- When Player 1 chooses NC, the best response for Player 2 (red) is C. When Player 1 chooses C, the best response for Player 2 is C
- **The (inefficient) pure-strategy NE of this game is thus (C, C).** Notice that we would obtain the same prediction using *IESDS* (C is dominant for both players)

Another Example

Player 2

Player 1		*Left*	*Middle*	*Right*
	Up	$(\underline{1}, 0)$	$(\underline{1}, \underline{2})$	$(0, 1)$
	Down	$(0, \underline{3})$	$(0, 1)$	$(\underline{2}, 0)$

The strategies (*Up*, *Middle*) represent the only *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy
- There is no strict incentive to *unilaterally deviate* from the equilibrium

NE in the Battle of Sexes

		Pat	
		Opera	Fight
Carl	Opera	$(\underline{2}, \underline{1})$	$(0, 0)$
	Fight	$(0, 0)$	$(\underline{1}, \underline{2})$

- The strategies (O, O) and (F, F) are both *pure-strategy NE* of this game
- Reasonable prediction as players get utility from being together

Relation between NE and IESDS

a. If *IESDS* eliminates **all but the strategies $(s_1^*, ..., s_n^*)$**, then these strategies are the only *NE* of the game;

b. If the strategies $(s_1^*, ..., s_n^*)$ are a *NE*, then they **survive IESDS**. On the other hand, there may be strategies that survive *IESDS* that are not part of any *NE*.

		Player 2	
		Left	*Middle*
Player 1	*Up*	$(1, 0)$	$(1, 2)$

- *NE* is a stronger solution concept than *IESDS*
- John Nash was able to show that there exists **at least one NE** in any game, possibly involving **mixed strategies**, that we present in the next slides

Fable of the Monks and Common Knowledge (1)

- There are n monks in a monastery. They cannot communicate with each other, but they gather once a week to pray:
  1. They see each other
  2. They do not see their own face
- There is a **disease** ravaging the land, whose first sign is a **mark** on the victim's forehead. Anyone who learns that he has contracted the disease immediately **commits suicide**
- One day a visitor announces: **at least one** of the monks in this room has the mark.
- After exactly **seven days**, all monks commit suicide.
- How many monks were there in the monastery? How many had a mark?

Fable of the Monks and Common Knowledge (2)

- Answer: the monks are seven and all of them had a mark
- Day 1: they all see each other. Each monk sees the mark on the others' forehead, but nobody commits suicide.
- Day 2: they all see each other. Each monk sees the mark on the others' forehead, but nobody commits suicide.

$$\vdots$$

- Day 7: they all commit suicide. This occurs as they can now be sure that they have the mark

Outline

Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: quantity of a *homogeneous* good to be produced (infinite is not included in the production interval)

$$S_i = S_j = [0, \infty) or$$
$$q_i = q_j = [0, \infty).$$

- Inverse market demand is $p(q) = a - Q$
- Payoffs: firm's profits

$$\pi_i = [a - Q - c]q_i,$$

where $Q = q_i + q_j$, c is constant *marginal cost* of production, and a is a parameter

Maximization Problem

- Each firm chooses the quantity that maximises its profits **given the quantity chosen by the other firm** (the other firm's **best-response**)

$$\max_{s_i} u_i(s_i, s_j^*) = \max_{0 \leq q_i < \infty} \pi_i(q_i, q_j^*),$$

$$\max_{s_j} u_j(s_i^*, s_j) = \max_{0 \leq q_j < \infty} \pi_j(q_i^*, q_j).$$

- Replace the equation of profits

$$\max_{q_i} [a - (q_i + q_j^*) - c] q_i,$$

$$\max_{q_j} [a - (q_i^* + q_j) - c] q_j.$$

Best Responses

- Maximisation is given by **first-order conditions**, obtained by taking the partial derivative of each firm's profit with respect to quantity produced and equating it to zero

$$\frac{\delta \pi_i(q_i, q_j^*)}{\delta q_i} = a - 2q_i - q_j^* - c = 0,$$
$$\frac{\delta \pi_j(q_i^*, q_j)}{\delta q_j} = a - 2q_j - q_i^* - c = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity

$$\begin{cases} q_i^* = \dfrac{a - q_j^* - c}{2} \\ q_j^* = \dfrac{a - q_i^* - c}{2} \end{cases}$$

Nash Equilibrium

- This is a system of 2 equations in 2 unknowns. *Pure-strategy NE* is obtained by replacing the second equation into the first

$$q_i^* = \frac{1}{2}\left[a - \frac{1}{2}\left(a - q_i^* - c\right) - c\right]$$

$$4q_i^* = 2a - a + q_i^* + c - 2c$$

$$3q_i^* = a - c$$

$$q_i^* = \frac{a-c}{3} = q_j^*$$

- Notice that we need $a > c$ in order for the equilibrium quantity to be positive

Equilibrium Profits

- To find equilibrium payoffs (profits), we plug equilibrium quantities in the firms' profit function

$$\pi_i^c(q_i^*, q_j^*) = \pi_j^c(q_i^*, q_j^*) = \left[a - \left(\frac{a-c}{3} + \frac{a-c}{3}\right) - c\right]\frac{a-c}{3} =$$

$$= \left[\frac{3a - 3c - 2a + 2c}{3}\right]\frac{a-c}{3} = \left(\frac{a-c}{3}\right)\left(\frac{a-c}{3}\right) =$$

$$= \frac{(a-c)^2}{9}$$

Monopoly vs Cournot (1)

- Recall that monopoly quantity and profits are given by

$$\max_q \pi(q) = [a - q - c]q$$

$$\frac{\delta\pi^m(q)}{\delta q} = a - 2q - c = 0,$$

$$q^m = \frac{a-c}{2} \ \ and \quad p_m = a - \frac{a-c}{2} = \frac{a+c}{2}.$$

$$\pi^m(q^m) = \frac{(a-c)^2}{4}.$$

- If firms could split the market *equally* the could produce half the monopoly quantity $\frac{a-c}{4}$ and get half the monopoly profit. This would be **larger than the Cournot profit**

Monopoly vs Cournot (2)

$$\frac{(a-c)^2}{8} > \frac{(a-c)^2}{9}$$

- However, notice that half the monopoly quantity **is not the best response** of one firm when the other produces half the monopoly quantity $\frac{q^m}{2} = \frac{a-c}{4}$

$$q_i^D = \frac{1}{2}\left[a - \frac{a-c}{4} - c\right]$$

$$8q_i^D = [4a - a + c - 4c]$$

$$q_i^D = \frac{3(a-c)}{8} > \frac{(a-c)}{4}.$$

Monopoly vs Cournot (3)

- Notice that this is larger than half the monopoly quantity. In other words, firms have an **incentive to produce more** as this **increases their profits**
- Define π^D firm $i's$ deviation profit

$$\pi_i^D = \left[a - \frac{3(a-c)}{8} - \frac{(a-c)}{4} - c\right]\frac{3(a-c)}{8} =$$

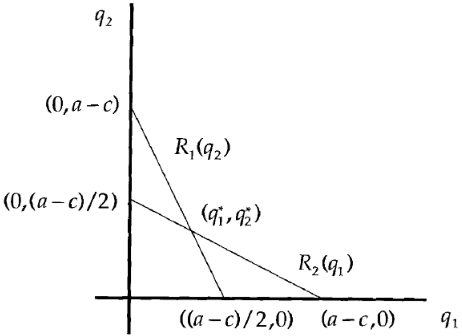
$$= \left[\frac{8a - 3a + 3c - 2a + 2c - 8c}{8}\right]\frac{3(a-c)}{8} =$$

$$= \left[\frac{3a - 3c}{8}\right]\frac{3(a-c)}{8} =$$

$$= \frac{3(a-c)}{8}\frac{3(a-c)}{8} =$$

$$= \frac{9}{64}(a-c)^2 > \frac{(a-c)^2}{8} = \frac{\pi^M}{2}$$

Graphical Interpretation Using Reaction Functions



Quantities are strategic substitutes:

$$\begin{cases} R_i(q_j) = \dfrac{a - q_j - c}{2} \\ R_j(q_i) = \dfrac{a - q_i - c}{2} \end{cases}$$

Remarks and Extensions

Remark:

- The game can be solved also using *IESDS* (not shown)

Extensions:

- Effect of including **fixed costs** F on *NE* equilibrium prices, quantities and profits?
- Effect of the two firms facing **different marginal costs** on *NE* equilibrium prices, quantities and profits?

Cournot with Fixed Costs (1)

- In this case profits are reduced by a **fixed cost** F

$$\pi_i = [a - Q - c]q_i - F,$$
$$\pi_j = [a - Q - c]q_j - F,$$

- Each firm maximises profits **given the quantity chosen by the other firm** (the other firm's **best-response**)

$$\max_{q_i}[a - (q_i + q_j^*) - c]q_i - F,$$
$$\max_{q_j}[a - (q_i^* + q_j) - c]q_j - F.$$

- When taking the first derivative, however, notice that F **cancels out** (it is a constant). F **does not affect** NE quantities. However, it **reduces** NE profits

Cournot with Asymmetric Costs

- In this case we assume that firm i is **more efficient** than j

$$c_j > c_i$$

- To solve the game, sufficient to follow all the steps provided for the general case. Not possible, however, to use the **general formula** to compute NE quantities.

- The more efficient firm will produce a **higher quantity** and make **larger profits**

Outline

Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: price of a *homogeneous* good to be produced (infinite is not included in the production interval)

$$S_i = S_j = [0, \infty) or$$
$$p_i = p_j = [0, \infty).$$

- Each firm's demand is (Q is market demand)

$$q_i = \begin{cases} 0 & if \;\; p_i > p_j \\ \frac{1}{2}Q & if \;\; p_i = p_j \\ Q & if \;\; p_i < p_j \end{cases}$$

- Payoffs: firm's profits (same marginal cost)

The Bertrand Paradox

The duopolists will set the **same price**, and this will be equal to their **marginal cost**. As a consequence, both firms will make zero profits. Let us see why (take ϵ very small number)

- $p_i > p_j > c$: Firm i sells nothing, it could serve all the market by setting $p_i = p_j - \epsilon$. **NO NE**;
- $p_i = p_j > c$: Firm i sells half the market quantity and makes a positive profit. However, it could set $p_i = p_j - \epsilon$ and serve the whole market **NO NE**;
- $c < p_i < p_j$: Firm i serves all the market, but it could increase its profits by increasing p_i to $p_i = p_j - \epsilon$ **NO NE**;
- $p_i < c$: The firm makes negative profits. **NO NE**.

The two firms will **undercut** each other up to the point in which

$$p_i = p_j = c$$

Differentiated Bertrand

- The way to solve the *paradox* is to assume that the two firms produce **differentiated goods**
- Each firm's demand is

$$q_i(p_i, p_j) = a - p_i + bp_j.$$

- $b > 0$ is a parameter that captures the **substitutability** between the two goods. When $\Delta p_j > 0$, $q_i > 0$. The effect increases as b becomes larger
- Firms' profits are

$$\pi_i(p_i, p_j) = [a - p_i + bp_j][p_i - c],$$
$$\pi_j(p_i, p_j) = [a - p_j + bp_i][p_j - c].$$

Maximisation Problem

- Each firm chooses the price that maximises its profits **given the price chosen by the other firm** (the other firm's **best-response**)

$$\max_{s_i} u_i(s_i, s_j^*) = \max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*),$$

$$\max_{s_j} u_j(s_i^*, s_j) = \max_{0 \leq p_j < \infty} \pi_j(p_i^*, p_j).$$

- Replace the equation of profits

$$\max_{p_i}[a - p_i + bp_j^*][p_i - c],$$

$$\max_{p_j}[a - p_j + bp_i^*][p_j - c].$$

Best Responses

- Maximisation is given by **first-order conditions**, obtained by taking the partial derivative of each firm's profit with respect to price and equating it to zero

$$\frac{\delta \pi_i(p_i, p_j^*)}{\delta p_i} = a - 2p_i + bp_j^* + c = 0,$$
$$\frac{\delta \pi_j(p_i^*, p_j)}{\delta p_j} = a - 2p_j + bp_j^* + c = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity

$$\begin{cases} p_i^* = \dfrac{a + bp_j^* + c}{2} \\ p_j^* = \dfrac{a + bp_i^* + c}{2} \end{cases}$$

Nash Equilibrium

- This is a system of 2 equations in 2 unknowns. *Pure-strategy NE* is obtained by replacing the second equation into the first

$$p_i^* = \frac{1}{2}\left[a + \frac{b}{2}\left(a + bp_i^* + c\right) + c\right]$$
$$4p_i^* = 2a + ba + b^2 p_i^* + bc + 2c$$
$$(4 - b^2)p_i^* = 2(a + c) + b(a + c)$$
$$(4 - b^2)p_i^* = (a + c)(2 + b)$$
$$p_i^* = \frac{(a + c)(2 + b)}{4 - b^2}$$
$$p_i^* = \frac{(a + c)(2 + b)}{(2 - b)(2 + b)}$$
$$p_i^* = \frac{a + c}{2 - b} = p_j^*$$

- Notice that we need $b < 2$ in order for the equilibrium price to be positive. If $b = 0$ the goods are not substitutes, and the duopolist can act as a monopolist

Equilibrium Profits

- To find equilibrium payoffs (profits), we plug equilibrium prices in the firms' profit function

$$\pi_i(p_i^*, p_j^*) = \pi_j(p_i^*, p_j^*) = \left[a - \frac{a+c}{2-b} + b\frac{a+c}{2-b}\right]\left[\frac{a+c}{2-b} - c\right] =$$

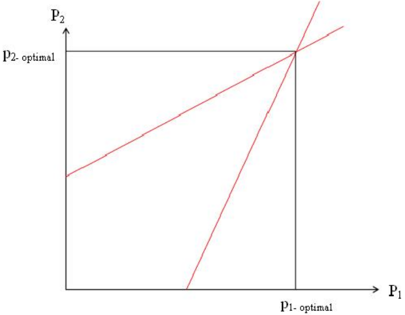
$$= \left[a + (b-1)\frac{a+c}{2-b}\right]\left[\frac{a+c}{2-b} - c\right] =$$

$$= \left[\frac{2a - ab + ab + bc - a - c}{2-b}\right]\left[\frac{a-c+bc}{2-b}\right] =$$

$$= \left[\frac{a-c+bc}{2-b}\right]\left[\frac{a-c+bc}{2-b}\right] = \left[\frac{a-c+bc}{2-b}\right]^2$$

- When $b = 0$, each duopolist makes monopoly profits

Graphical Interpretation Using Reaction Functions



Prices are strategic complements:

$$\begin{cases} R_i(p_j) = \dfrac{a + bp_j^* + c}{2} \\ R_j(p_i) = \dfrac{a + bp_i^* + c}{2} \end{cases}$$

Remarks and Extensions

Remark:

- The game can be solved also using *IESDS* (not shown)

Extensions:

- Effect of including **fixed costs** F on *NE* equilibrium prices, quantities and profits? (same as in Cournot)
- Effect of the two firms facing **different marginal costs** on *NE* equilibrium prices, quantities and profits? (same as in Cournot)

Outline

"If citizens respond only to private incentives, public goods will be **underprovided** *and public resources* **overutilised**.*"*

David Hume (1739)

Rules of the Game (1)

- **Players**: n farmers in a village graze goats on the common green
- **Strategies**: number of goats to own $[0, G_max]$
  - The *ith* farmer owns g_i goats
  - The total number of goats in the village is

$$G = g_1 + ... + g_i + ... + g_n$$

- Farmer choose the number of goats to own *simultaneously*

Rules of the Game (2)

- **Payoffs**: value of grazing a goat minus its cost
  - Value is positive only if $G < G_{max}$, the *maximum amount* of goats that can survive on the green

$$\begin{cases} V(G) > 0 \;\; if \;\; G < G_{max} \\ V(G) = 0 \;\; if \;\; G > G_{max} \end{cases}$$

  - *Marginal effect* of adding a goat is *negative* on $V(G)$. The negative effect is *increasing* in the number of goats

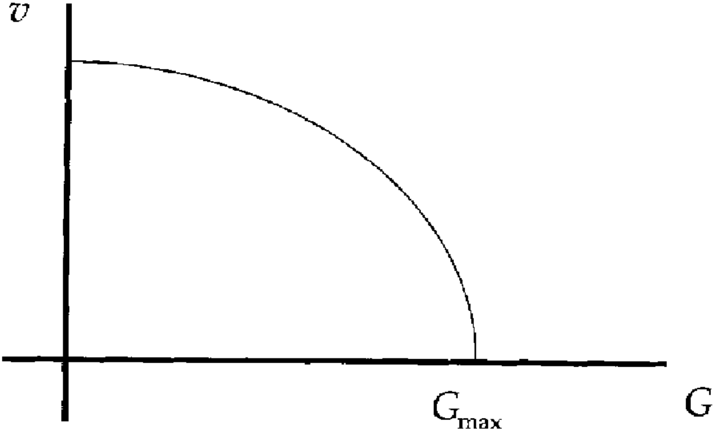
$$\frac{\delta V(G)}{\delta G} < 0.$$

$$\frac{\delta^2 V(G)}{\delta G^2} < 0.$$

  - Payoffs are

$$g_i V(G) - cg_i.$$

Decreasing and Diminishing Marginal Returns



Individual Maximisation Problem

- Farmer i maximises its payoffs by the number of goats **given** that the other farmer are playing their equilibrium strategies

$$\max_{g_i} \; g_i V(g^*_{-i}, g_i) - c g_i.$$

- We take the *first derivative* wrt g_i and equate it to zero

$$V(g^*_{-i}, g_i) + g_i V^{'}(g^*_{-i}, g_i) - c = 0.$$

- As the solution is *symmetric* (farmers have the same strategies and payoff functions) we can replace $g_i = \frac{G^*}{n}$

$$V(G^*) + \frac{G^*}{n} V^{'}(G^*) - c = 0.$$

Maximisation by Social Planner

- In contrast, the *social planner* (assumed to be benevolent), would maximise **social** benefits from grazing

$$\max_{0 \leq G \leq \infty} \quad GV(G) - cG.$$

- We take the *first derivative* wrt G and equate it to zero

$$V(G^{**}) + G^{**}V^{'}(G^{**}) - c = 0.$$

Comparison of Equilibria

- As both *first-order conditions* equal zero, the following equality must hold

$$V(G^*) + \frac{G^*}{n} V^{'}(G^*) = V(G^{**}) + G^{**} V^{'}(G^{**}).$$

- We claim that $G^* > G^{**}$. To prove this we use **contradiction**, i.e. we assume $G^* < G^{**}$. This implies
  1. $V(G^*) > V(G^{**})$ since $V' < 0$
  2. $V'(G^*) > V'(G^{**})$ since $V'' < 0$
  3. $\frac{G^*}{n} < G^{**}$ since $G^* < G^{**}$
- As a consequence, the left-hand side of the equation is **strictly larger**!
- This contradicts the equality, so it must be $G^* > G^{**}$. Farmers graze **too many goats** because each farmer considers only his own incentives

Outline

Matching Pennies Revisited

		Player 2	
		H	T
Player 1	H	$(\underline{1}, -1)$	$(-1, \underline{1})$
	T	$(-1, \underline{1})$	$(\underline{1}, -1)$

- The concept of *NE* (in pure strategies) fails in these games
- One player would like to **outguess** the other. Examples are
  - Poker $\rightarrow$ How often to bluff?
  - Battle $\rightarrow$ Attack by land or sea?
- No *pure-strategy NE* exists in these games, as there is **uncertainty** concerning what the other players will do
- Uncertainty is captured by the concept of **mixed strategy**

Mixed Strategies

Definition (*mixed strategy*):
In the normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$ *suppose* $S_i = \{s_{i1}, ..., s_{iK}\}$ *(*s_{ik} *is a generic pure strategy* k *for player* i*).*
Then a **mixed strategy** *for player* i *is a* **probability distribution** $p_i = (p_{i1}, ..., p_{iK})$*, where* $0 \leq p_{1k} \leq 1$ *for* $k = 1, ...K$ *and* $p_{i1} + ... + p_{iK} = 1$.

- We denote the game accounting for mixed strategies as

$$\Gamma = \{N, P_i(S_i)_{i \in N}, \nu_{i \in N}\}$$

- If the probability attached to s_{ik} is $p_{ik} = 1$, then the player is playing a pure strategy

Mixed-Strategy NE (1)

- Let $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{ik}, ..., \sigma_{iK})$ the set of all mixed strategies available to player i
- Let $\sigma_{ik} = p_{ik} s_{ik}$ be a generic mixed strategy for player i with $\sum_{k=1}^{K} p_{ik} = 1,\ \ p_{ik} \geq 0$

Definition (*NE in mixed strategies*):
In the normal-form game $\Gamma = \{N, P_i(S_i)_{i \in N}, \nu_{i \in N}\}$ a mixed strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_N^*)$ constitutes a *NE* if for every $i = 1, ...N$

$$\nu_i(\sigma_i^*, \sigma_{-i}^*) \leq \nu_i(\sigma_i^{'}, \sigma_{-i}^*)$$

for every $\sigma_i^{'} \in P_i(S_i)$

Mixed-Strategy NE (2)

- Mixed-strategy *NE* represent uncertainty by one player with respect to what the other player will do
- This is why they choose to **randomise** over their pure strategies
- Simply an extension of pure-strategy *NE*, as it requires that a mixed strategy is a **best-response** to the opponent's equilibrium mixed strategy

Caution: there exists mixed-strategy equilibria where best responses are played with probability 1. These are pure-strategy NE

Mixed-Strategy Equilibrium in Matching Pennies (1)

- Let $0 \leq x \geq 1$ and $0 \leq y \leq 1$ be the probabilities that player 1 and 2 choose *Head*

			Player 2	
			y	1-y
			H	T
Player 1	x	H	$(1, -1)$	$(-1, 1)$
	1-x	T	$(-1, 1)$	$(1, -1)$

- We divide this problem in 6 cases:
  1. $x = 1$, player 1 plays H with certainty $\rightarrow$ No equilibrium
  2. $x = 0$, player 1 plays T with certainty $\rightarrow$ No equilibrium
  3. $x \in (0, 1) \rightarrow$ Player 1 randomises

  1. $y = 1$, player 2 plays H with certainty $\rightarrow$ No equilibrium
  2. $y = 0$, player 2 plays T with certainty $\rightarrow$ No equilibrium
  3. $y \in (0, 1) \rightarrow$ Player 2 randomises

Mixed-Strategy Equilibrium in Matching Pennies (2)

- In order for a *mixed-strategy NE* to arise, we need
  1. Player 1 have the same expected payoff (payoff times probability) from playing H or T when player 2 plays the mixed strategy y, i.e.

     $$\nu_1(H, y) = \nu_1(T, y)$$

  2. Player 2 have the same expected payoff (payoff times probability) from playing H or T when player 2 plays the mixed strategy x, i.e.

     $$\nu_2(x, H) = \nu_2(x, T)$$

- If these conditions are not satisfied, players will prefer some pure strategy, as shown below

Mixed-Strategy Equilibrium in Matching Pennies (2)

- Take the expected payoffs of player 1

$$\nu_1(H, y) = 1 \times y + (-1) \times (1 - y) = 2y - 1,$$
$$\nu_1(T, y) = (-1) \times y + 1 \times (1 - y) = 1 - 2y.$$

- Let us suppose $\nu_1(H, y) > \nu_1(T, y)$. Then

$$2y - 1 > 1 - 2y \implies 4y > 2 \implies y > \frac{1}{2}.$$

  If $y > 1/2$ player 1 chooses H with certainty

- Let us suppose instead $\nu_1(H, y) < \nu_1(T, y)$. Then

$$2y - 1 < 1 - 2y \implies 4y < 2 \implies y < \frac{1}{2}.$$

  If $y > 1/2$ player 1 chooses T with certainty

Mixed-Strategy Equilibrium in Matching Pennies (3)

- Take the expected payoffs of player 2

$$\nu_2(x, H) = (-1) \times x + 1 \times (1 - x) = 1 - 2x,$$
$$\nu_2(x, T) = 1 \times x + (-1) \times (1 - x) = 2x - 1.$$

- Let us suppose $\nu_2(x, H) > \nu_2(x, T)$. Then

$$1 - 2x > 2x - 1 \implies 4x < 2 \implies x < \frac{1}{2}.$$

  If $x < 1/2$ player 2 chooses H with certainty

- Let us suppose instead $\nu_2(x, H) < \nu_2(x, T)$. Then

$$1 - 2x < 2x - 1 \implies 4x > 2 \implies x > \frac{1}{2}.$$

  If $x > 1/2$ player 1 chooses T with certainty

Mixed-Strategy Equilibrium in Matching Pennies (3)

- The only probabilities such that Players 1 and 2 are willing to randomise over their pure strategies are respectively $y = 1/2$ and $x = 1/2$
- There is only 1 *NE* in this game, it is in mixed strategies, and can be denoted as

$$\left(\frac{1}{2}, \frac{1}{2}\right) \text{ or } \left(\frac{1}{2}H + \frac{1}{2}T; \frac{1}{2}H + \frac{1}{2}T\right)$$

Mixed-Strategy Equilibrium in the Battle of Sexes (1)

			Pat	
			y	1-y
			Opera	Fight
Carl	x	Opera	$(2, 1)$	$(0, 0)$
	1-x	Fight	$(0, 0)$	$(1, 2)$

- In this case we have two *pure-strategy NE*. They occur when

$$x = 1, y = 1 \text{ and } x = 0, y = 0.$$

- Are there any *mixed-strategy NE*?

Mixed-Strategy Equilibrium in the Battle of Sexes (2)

- Take the expected payoffs for Carl

$$\nu_C(O, y) = 2 \times y + (0) \times (1 - y) = 2y,$$
$$\nu_C(T, y) = 0 \times y + 1 \times (1 - y) = 1 - y.$$
$$2y = 1 - y \implies y = 1/3.$$

- Take the expected payoffs for Pat

$$\nu_C(x, O) = 1 \times x + (0) \times (1 - x) = x,$$
$$\nu_C(x, T) = 0 \times x + 2 \times (1 - x) = 2 - 2x.$$
$$x = 2 - 2x \implies x = 2/3.$$

- The *mixed-strategy NE* is

$$\left(\frac{1}{3}, \frac{2}{3}\right) \text{or} \left(\frac{2}{3} O + \frac{1}{3} F; \frac{1}{3} O + \frac{2}{3} F\right)$$

Existence of Nash Equilibrium

Theorem *(Nash, 1950): In the n-player normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$, if n is finite and S_i is finite for every i then there exists at least one Nash equilibrium, possibly involving mixed strategies.*