

# Games, Information and Contract Theory

## Problem Set 1

Instructors

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*This problem set will be solved in class on Monday, September 23, 2019.*

### Exercise 1.

Consider the static game of complete information described by the following payoff matrix:

		Player 2		
		L	C	R
Player 1	T	(2,0)	(1,1)	(4,2)
	M	(3,4)	(1,2)	(2,3)
	B	(1,3)	(0,2)	(3,0)

**Answer the following questions and explain your answers in detail.**

1. Write the game as a normal-form game (*hint: write separately the set of players  $N = \{\}$ , the set of strategies available to each player  $S_i = \{\}$ , and the payoffs associated to each outcome of the game for each player  $u_i(s_i, s_j)$ . Then write  $G$  (no need to write all the payoffs in this case, bi-matrix is enough).*)
2. Apply *Iterated Elimination of Strictly Dominated Strategies (IESDS)*, and write the strategies that survive this process (*hint: check for the presence of a strictly dominated strategy for Player 1, then move to Player 2. Iterate*).
3. Find the *pure-strategy Nash equilibria (NE)* of this game (*hint: find the players' best response to their opponent's strategy and show that there is no profitable deviation*).

- Are your findings in points 2 and 3 different? Explain the relation between *IESDS* and *NE*.

### Exercise 2. (Study together)

Ann and Paul have to study for their Game Theory exam. They can decide to study at their own home or at the university library. If they both remain at home, Ann's payoff is 2 and Paul's payoff is 0; if they both study at the university, they study together and Ann's payoff is equal to Paul's payoff; if Ann goes to the library and Paul does not, Ann's payoff is 1 and Paul's payoff is -1; otherwise, if Ann remains at home and Paul goes to the library, Ann's payoff is 2 and Paul's payoff is 1.

*Hint: the payoff from Library-Library can be written as  $(x, x)$ , two equal numbers.*

		Paul	
		Home	Library
Ann	Home	( , )	( , )
	Library	( , )	( $x, x$ )

**Answer the following questions and explain your answers in detail.**

- Fill the payoff matrix using the available information.
- Write the game as a normal-form game.
- Can 'study at the library' be a dominant strategy for Ann? Can 'study at home' be a dominant strategy for Ann? And for Paul? Explain (*hint: this depends on the values taken by  $x$* ).
- Find the minimum payoffs for Ann and Paul in case they both go to the library such that 'both going to the library' is a Nash equilibrium (NE). Is there any other NE? (*hint: playing Library must be a best response for Ann when Paul plays Library, and vice-versa.*)

### Exercise 3. (Cournot Duopoly)

Two firms compete in a market by simultaneously setting the quantities of a (homogeneous) good to produce  $(q_i, q_j)$ . Each firm faces a *constant marginal cost*  $c = 3$ . The two firms face the *inverse demand function*  $P(Q) = 9 - Q$ , where  $Q$  is the aggregate quantity produced. Payoffs are given by each firm's profits.

**Answer the following questions and explain your answers in detail.**

- Describe the game as a normal-form game (*Players, Strategies, Payoffs*).
- Write the maximisation problem for each firm.

3. Solve the maximisation problem for each firm, obtaining its *reaction function*. (*hint: take the first derivative of profits with respect to each firm's quantity, taking the quantity produced by the other as given*).
4. Find the *NE* of this game (equilibrium quantities for both firms) (*hint: use the two reaction functions and solve for  $q_i^*$* ).
5. Find the payoffs ( $\pi_i(q_i^*, q_j^*)$ ) obtained by firms when they play this *NE* (*hint: replace equilibrium quantities in profits*).
6. Now suppose that both firms also face a *fixed cost*  $F = 1$ . Does this affect the equilibrium quantity? Does this change equilibrium payoffs? Explain (*hint: solve again the maximisation problem, also including  $F$* ).

#### Exercise 4. (Bertrand Duopoly)

Two firms compete in a market by simultaneously setting the prices of a differentiated good  $(p_i, p_j)$ . Each firm faces a *constant marginal cost*  $c = 2$ . The demand for each firm's good is  $q_i(p_i, p_j) = 6 - p_i + bp_j$  and  $q_j(p_i, p_j) = 6 - p_j + bp_i$ , where  $b$  is a parameter capturing product differentiation. We set  $b = 1$ . Payoffs are given by each firm's profits.

1. Describe the game as a normal-form game (*Players, Strategies, Payoffs*).
2. Write the maximisation problem for each firm.
3. Solve the maximisation problem for each firm, obtaining its *reaction function*. (*hint: take the first derivative of profits with respect to each firm's price, taking the price set by the other as given*).
4. Find the *NE* of this game (equilibrium prices for both firms) (*hint: use the two reaction functions and solve for  $p_i^*$* ).
5. Find the payoffs ( $\pi_i(p_i^*, p_j^*)$ ) obtained by firms when they play this *NE* (*hint: replace equilibrium prices in profits*).

## Problem of the commons

Here follows a short explanation of the "Problem of the Commons" presented in class.

In a village,  $n$  farmers have to choose simultaneously how many goats to graze on the village green. Denote the number of goats owned by the  $i$ th farmer by  $g_i$ . The total number of goats in the village is thus

$$G = g_1 + \dots + g_i + \dots + g_n \quad (1)$$

Farmers pay a unit cost  $c$  to purchase and feed each goat. The *value* of owning a goat when  $G$  goats are grazing is  $v(G)$  *per goat*. Also assume that the value of owning a goat becomes zero if too many goats are grazing the green.

$$v(G) \begin{cases} > 0, & \text{if } G < G_{max} \\ 0, & \text{if } G \geq G_{max} \end{cases} \quad (2)$$

Moreover, we assume that the value of grazing each goat decreases in the total number of goats (first derivative). Moreover, the value drop is larger when more goats are already grazing (second derivative). Formally:

$$v'(G) < 0 \quad (3)$$

$$v''(G) < 0 \quad (4)$$

Each farmer solves the following maximisation problem (taking the number of goats chosen by the other farmers as given):

$$\max_{g_i} g_i v(g_1^* + \dots + g_{i-1}^* + g_i + g_{i+1}^* + \dots + g_n^*) - c g_i$$

or, alternatively

$$\max_{g_i} g_i v(G) - c g_i \quad (5)$$

Taking the first derivative and equating it to zero yields

$$v(G^*) + g_i v'(G^*) - c = 0 \quad (6)$$

where  $G^*$  denotes the quantity which satisfies (6). Provided that this is a *symmetric*

problem (i.e. farmers share the same payoffs and costs) this can be written as

$$v(G^*) + \frac{G^*}{n}v'(G^*) - c = 0 \quad (7)$$

We now want to show that this quantity is larger than the one chosen by the benevolent *social planner*, whose is interested in maximising the overall payoff for the society, i.e. the total value of goats minus their cost. The latter solves

$$\max_G Gv(G) - cG \quad (8)$$

Taking the first derivative and equating it to zero yields

$$v(G^{**}) + G^{**}v'(G^{**}) - c = 0 \quad (9)$$

where  $G^{**}$  denotes the quantity which satisfies (9). We now show that  $G^* > G^{**}$ , i.e. that farmers, by choosing independently, end up with a too large amount of goats. We use *proof by contradiction*, i.e. we assume the opposite, i.e.  $G^* < G^{**}$ , and show that this would imply a contradiction.

Assume  $G^* < G^{**}$ .

As both 6 and 9 equal zero, it is true that

$$v(G^*) + \frac{G^*}{n}v'(G^*) = v(G^{**}) + G^{**}v'(G^{**}) \quad (10)$$

- $v(G^*)$  and  $v(G^{**})$  are positive. Given the previous assumption,  $v(G^*) > v(G^{**})$ .
- $\frac{G^*}{n}$  and  $G^{**}$  are positive. Given the previous assumption,  $\frac{G^*}{n} < G^{**}$ .
- $v'(G^*)$  and  $v'(G^{**})$  are negative and, by the previous assumption,  $v'(G^*) > v'(G^{**})$  (or, more precisely,  $v'(G^*)$  is *less negative*).
- This implies  $\frac{G^*}{n}v'(G^*) > G^{**}v'(G^{**})$ , in the sense that the first is less negative than the second.

The left-hand side of the equation would thus be larger than the right-hand side. But this would contradict the equality between (6) and (9). Assuming  $G^* < G^{**}$  thus leads to a contradiction, i.e. it must be that  $G^* > G^{**}$ .

The number of goats grazing in the Nash Equilibrium is thus too large when compared to the social optimum.