Repeated Games

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Outline

Issue Addressed

- Can **threats** about **future behaviours**...
- ...affect **current behaviours** in **repeated relationships**?
- In other words, is it possible to achieve a **cooperative outcome** in **repeated interactions**?

Assumptions

- A **repeated game** is the same **static game of complete information** played more than once
- Two possible **cases** of repeated games:
  1. **Finitely**: the game ends after a **known number of repetitions**
  2. **Infinitely**: the game does not have an end. Alternatively, the players **do not know** when it ends
- **Equilibrium concept**: *SPNE*. Solution in cases (1) and (2) will differ substantially
- We start with **finitely repeated** games

Outline

Stage Game - Definition

Let $G = \{A_1, ..., A_N; u_1, ..., u_n\}$ *denote a static game of complete information in which players 1 through n* **simultaneously** *choose* **actions** a_1 *through* a_n *from the* **action spaces** A_1 *through* A_n*, respectively, and* **payoffs** *are* $u_1(a_1, ..., a_n)$ *through* $u_n(a_1, ..., a_n)$*. The game G will be called the* **stage game** *of the repeated game.*

- Notice that the definition mentions **actions**, not strategies
- **Strategies** specify what players can do at each of their decision nodes (this is a dynamic game of imperfect info)

Finitely Repeated Games and SPNE

Definition: *Given a* **stage game** G, *let* $G(T)$ *denote the* **finitely repeated game** *in which* G *is played* T *times, with the outcomes of all preceding plays observed before the next play begins. The* **payoffs** *for* $G(T)$ *are simply the* **sum of the payoff** *from the* T *stage games.*

Proposition: *If the stage game* G *has a* **unique Nash Equilibrium** *then, for any finite* T, *the repeated game G(T) has a* **unique subgame-perfect outcome**: *the* **Nash equilibrium** *of* G *is* **played in every stage**.

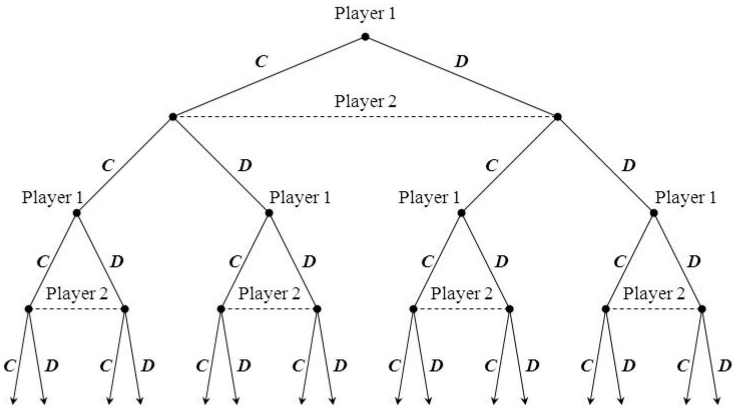
Two-Period Prisoners' Dilemma (1)

- Suppose the following modified version of the **Prisoners' Dilemma** is played **twice**. The **stage game** is

Prisoner 2 (columns), Prisoner 1 (rows)

	L_2	R_2
L_1	$(1, 1)$	$(5, 0)$
R_1	$(0, 5)$	$(4, 4)$

Two-Period Prisoners' Dilemma -Extensive Form (2)



Two-Period Prisoners' Dilemma (3)

- We start from the **second stage**

Prisoner 2

Prisoner 1		L_2	R_2
	L_1	$(1, 1)$	$(5, 0)$
	R_1	$(0, 5)$	$(4, 4)$

- The **only NE** in the second stage is (L_1, L_2). We can replace the payoffs from the second stage into the first stage

Two-Period Prisoners' Dilemma (4)

Prisoner 2

Prisoner 1	L_2	R_2
L_1	$(2, 2)$	$(6, 1)$
R_1	$(1, 6)$	$(5, 5)$

- We add the *NE payoffs* from stage 2 to the payoffs of stage 1
- The **only NE** of the first stage is again (L_1, L_2)
- The **SPNE** of this game is $((L_1, L_1), (L_2, L_2))$

An Artificial Mechanism (1)

- We **add the strategies** R_1 and R_2 to the Prisoners' dilemma

Prisoner 2

Prisoner 1		L_2	M_2	R_2
	L_1	$(1, 1)$	$(5, 0)$	$(0, 0)$
	M_1	$(0, 5)$	$(4, 4)$	$(0, 0)$
	R_1	$(0, 0)$	$(0, 0)$	$(3, 3)$

- **Two NE** of the stage game are (L_1, L_2) and (R_1, R_2)
- Suppose players anticipate that:
  - (R_1, R_2) will be played in stage 2 if (M_1, M_2) is played in 1
  - (L_1, L_2) will be played in stage 2 in any other case
- We can **add the NE payoffs** to the first stage

An Artificial Mechanism (2)

Prisoner 2

Prisoner 1	L_2	M_2	R_2
L_1	$(2, 2)$	$(6, 1)$	$(1, 1)$
M_1	$(1, 6)$	$(7, 7)$	$(1, 1)$
R_1	$(1, 1)$	$(1, 1)$	$(4, 4)$

- **Three NE** are (L_1, L_2), (M_1, M_2), and (R_1, R_2)
  - (R_1, R_2) in stage 1 corresponds to the *SP outcome* $((R_1, R_2), (L_1, L_2))$
  - (L_1, L_2) in stage 1 corresponds to the *SP outcome* $((L_1, L_2), (L_1, L_2))$
  - (M_1, M_2) in stage 1 corresponds to the *SP outcome* $((M_1, M_2), (R_1, R_2))$
- Cooperation is **possible**. However, is retaliation **credible**?

An Artificial Mechanism (3)

Problem is the possibility of **renegotiation** (bygones are bygones)

- Suppose that (M_1, M_2) **is not played** in stage 1
- As there are **two NE** in stage 2, better to play (R_1, R_2)
- This would achieve a payoff of $(3 > 1)$

The incentive to play (M_1, M_2) in the first stage is **destroyed**

- There are ways to solve the renegotiation problem in finitely repeated games (not shown)

Outline

The Discount Factor (1)

- The stage game is repeated an **infinite number of time**
- Measure of payoffs is the **present value** of the infinite sequence of payoffs computed using the **discount factor**

$$\delta = 1/(1+r),$$

where r is the interest rate. Notice that $0 < \delta < 1$

- The discount factor is the value **today** of a dollar to be received **one stage later**

Definition: *Given the discount factor δ, the* **present value** *of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, ...$ is*

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + ... = \sum_{t=1}^{T} \delta^{t-1}\pi_t.$$

The Discount Factor (2)

The discount factor can also be interpreted as the **probability** that the game ends

- Suppose that the game ends with probability p
- Then the expected payoff from the next stage is

$$\pi = (1 - p)\pi/(1 + r),$$

  and the one received two stages from now is

$$\pi = (1 - p)^2\pi/(1 + r)^2.$$

- The discount factor in this case is $\delta = (1 - p)/(1 + r)$

Definition of Infinitely Repeated Game

Definition:

Given a stage game G, let $G(\infty, \delta)$ denote the **infinitely repeated game** *in which G is repeated* **forever** *and the players share the* **discount factor** δ*. For each t, the outcomes of $t-1$ preceding plays of the stage game are* **observed** *before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the* **present value** *of the player's payoffs from the infinite sequence of stage games*

Strategies in Repeated Game

Definition*:*
In the finitely repeated game $G(T)$ or the infinitely repeated game $G(\infty, \delta)$, a player's **strategy** *specifies the action the player will take in* **each stage***, for each possible* **history of play** *through the previous stage*

- The **history of the play** through the stage t is the record of the players' choices (actions) in stages 1 through t
- For example, players might have chosen generic actions in stage s

$$(a_{11}, ..., a_{n1}) \text{ in stage 1,}$$

$$(a_{12}, ..., a_{n2}) \text{ in stage 2,}$$

$$(a_{1t}, ..., a_{nt}) \text{ in stage t,}$$

with $a_{is} \in A_i$

History of the Play in Prisoners' Dilemma

- Suppose the Prisoners' Dilemma is played **three times**

Prisoner 2

Prisoner 1		L_2	R_2
	L_1	$(1,1)$	$(5,0)$
	R_1	$(0,5)$	$(4,4)$

- Possible histories of the game at stage 3 are:
  - $((L_1,L_2),(L_1,L_2))$
  - $((L_1,L_2),(L_1,R_2))$
  - $((L_1,L_2),(R_1,R_2))$
  - $((L_1,L_2),(R_1,L_2))$
  - $((R_1,L_2),(L_1,L_2))$
  - $((L_1,R_2),(L_1,L_2))$
  - $((R_1,L_2),(R_1,L_2))$
  - $((R_1,L_2),(L_1,R_2))$
  - $((R_1,R_2),(L_1,L_2))$
  - $((R_1,R_2),(L_1,R_2))$
  - $((R_1,R_2),(R_1,R_2))$
  - $((R_1,R_2),(R_1,L_2))$

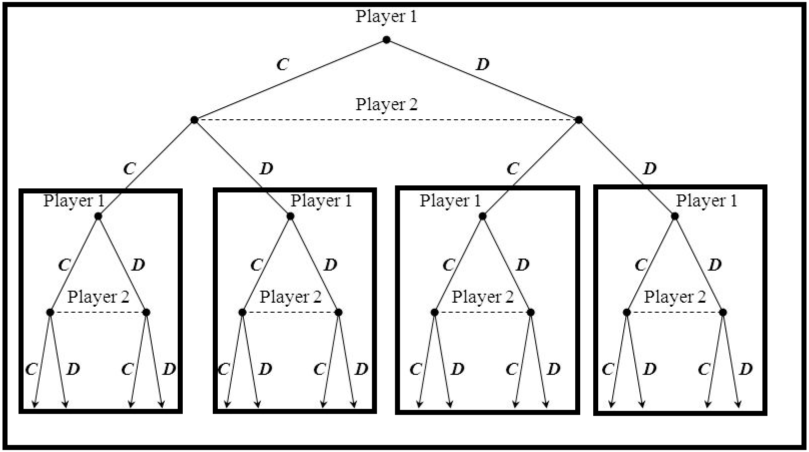
  etc...

Subgames in Repeated Game

Definition:

- *In the finitely repeated game $G(T)$, a* **subgame** *beginning at stage $t+1$ is the repeated game in which G is played $T-t$ times, denoted $G(T-t)$. There are many subgames that begin at stage $t+1$, one for each of the possible histories of play through stage t.*
- *In the infinitely repeated game $G(\infty, \delta)$, each* **subgame** *beginning at stage $t+1$ is identical to the original game $G(\infty, \delta)$. As in the finite-horizon case, there are as many subgames beginning at stage $t+1$ of $G(\infty, \delta)$ as there are possible histories of play through stage t.*

Subgames in Two-Period Prisoners' Dilemma



One subgame for each of the possible histories of play through t

Trigger Strategy and SPNE

- We look for a **cooperative NE** of infinitely repeated games
- We claim that the equilibrium based on the following **trigger strategy** for player *i* constitute (i) a *NE* of the infinitely repeated game and that (ii) this equilibrium is *subgame-perfect*:

  "Cooperate at stage t as long as the other players cooperated at stage $t-1$. Otherwise, play the NE of the stage game."

- In other words, player *i* **pulls the trigger** if he observes, in stage t, deviation by the other players from the cooperative outcome at $t-1$
- Player 1 **retaliates** if she observes deviations

Trigger Strategy - NE of the Whole game (1)

Compare the payoffs from **cooperation** and **deviation** in the Prisoners' Dilemma

- If player 1 **cooperates** in stage t when the other one plays the trigger strategy she gets

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + ... =$$

$$= 4(1 + \delta + \delta^2 + \delta^3 + ...) =$$

This is a **geometric series** converging to

$$4(1 + \delta + \delta^2 + \delta^3 + ...) = \frac{4}{1 - \delta},$$

since $\delta \leq 1$

Trigger Strategy - NE of the Whole game (2)

- If player 1 **deviates** when the other one plays the trigger strategy she gets

$$5 + 1\delta + 1\delta^2 + 1\delta^3 + ... =$$

$$= 5 + 1(\delta + \delta^2 + \delta^3 + ...) =$$

This is a **geometric series** converging to

$$5 + 1(\delta + \delta^2 + \delta^3 + ...) = 5 + \frac{\delta}{1-\delta},$$

since $\delta \leq 1$

Trigger Strategy - NE of the Whole game (3)

- In order for cooperation (R_1, R_2) to be achieved, the present value of cooperation payoffs must not be smaller than the ones from deviating

$$\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}.$$

Rearranging yields

$$4 \geq 5(1-\delta) + \delta \implies 4 \geq 5 - 5\delta + \delta \implies$$

$$\implies 4\delta \geq 1 \implies \delta \geq \frac{1}{4}$$

- The **trigger strategy** is a *NE* of the infinitely repeated game if players are **patient enough** $(\delta \geq 1/4)$
- The cooperative outcome can be achieved

Trigger Strategy - NE in all Subgames

- We have to show that the trigger-strategy equilibrium is **subgame-perfect**, i.e. that it is a *NE* in all the subgames

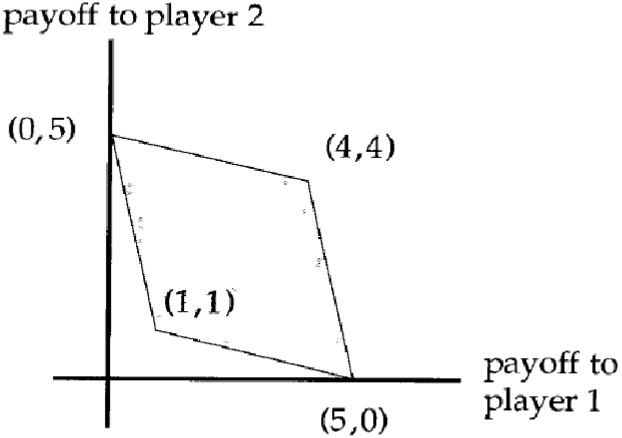
- Two types of subgames:
  i. The outcome of all earlier stages is (R_1, R_2)
  ii. The outcome of at least one earlier stage differs from (R_1, R_2)

- If player adopts the trigger strategy for the game as a whole:
  - In games belonging to *i*., her strategy is still the **trigger strategy**, which has been shown to be a *NE* equilibrium of the whole game if $\delta \geq 1/4$
  - In games belonging to *ii*., she plays the **NE of the stage game**, which is also a *NE* of the game as a whole if $\delta \leq 1/4$

Theory: Feasible Payoffs

- Let $(x_1, ..., x_n)$ be **feasible payoffs** in the stage game G
- **Feasible** means that they can be obtained as a **convex combination** (weighted average) of the stage-game payoffs
- Notice that weights are between 0 and 1 (convex)
- In the Prisoners' dilemma, feasible payoffs include
  1. Pairs (x, x) from averaging $(1, 1)$ and $(4, 4)$, for $1 < x < 4$
     e.g. $1(0.5) + 4(0.5) = 2.5$ or $1(0.7) + 4(0.3) = 1.9$.
  2. Pairs (y, z) from averaging $(0, 5)$ and $(5, 0)$, for $1 < y < 5$ and $y + z = 5$
     e.g. $0.5(0) + 0.5(5) = 2.5$ or $0.3(0) + 0.7(5) = 3.5$

Feasible Payoffs in Prisoners' Dilemma



All the payoffs that are part of the trapezoid are feasible

Theory: Average Payoffs

In order to allow for the comparison of feasible payoffs and stage-game payoffs, we define **average payoffs**

Definition: *Given the discount factor δ, the* **average payoff** *of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is*

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\pi_t$$

- For instance, in the Prisoners' cooperation could achieve

$$\frac{4}{1-\delta},$$

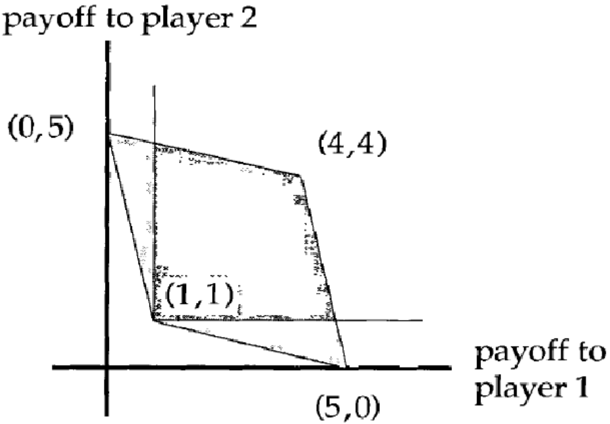
or, in terms of average payoffs, 4

Theory: The Folk's Theorem

Theorem *(Friedman 1971):*

Let G be a finite, static game of complete information. Let $(e_1, ...e_n)$ denote the **payoffs** *from a NE of G, and let $(x_1, ...x_n)$ denote any other* **feasible payoffs** *from G. If $x_i > e_i$* **for every player** *i and δ is sufficiently close to one, then there exists a subgame-perfect Nash Equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, ...x_n)$ as the* **average payoff**

The Folk's Theorem in Prisoners' Dilemma



NE payoffs are $(e_1, e_2) = (1, 1)$. All **feasible average payoffs** above and to the right of (e_1, e_2) can be achieved

Outline

Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: price of a *homogeneous* good to be produced

$$p_i = p_j = [0, \infty).$$

- Each firm's demand is (Q is market demand)

$$q_i = \begin{cases} 0 & if \quad p_i > p_j \\ \frac{1}{2}Q & if \quad p_i = p_j \\ Q & if \quad p_i < p_j \end{cases}$$

- Payoffs: firm's profits (same marginal cost)
- **The Firms set prices an infinite number of times**

Cooperation vs Deviation (1)

- All payoffs between 0 and monopoly profit are **feasible**
- If firms collude and set the **monopoly price**, each sells **half the monopoly quantity** and gets **half the monopoly profit**
- The two Firms play the **trigger strategy**:

"Set the monopoly price at stage t as long as the other Firm set the monopoly price at stage $t-1$. Otherwise, set price equal to marginal cost."

Cooperation vs Deviation (2)

- The **present value of cooperation** for Firm i (or j) is

$$PV^{coop} = \frac{\pi^m}{2} + \delta\frac{\pi^m}{2} + \delta^2\frac{\pi^m}{2} + \ldots = \frac{1}{1-\delta}\frac{\pi^m}{2}.$$

- The **present value of deviation** for Firm i (or j) is

$$PV^{dev} = \pi^m + \delta 0 + \delta^2 0 + \ldots = \pi^m.$$

- Cooperation can be achieved if

$$PV^{coop} \geq PV^{dev},$$

$$\frac{1}{1-\delta}\frac{\pi^m}{2} \geq \pi^m \implies \frac{1}{1-\delta}\frac{1}{2} \geq 1 \implies$$

$$\frac{1}{2} \geq 1-\delta \implies \delta \geq \frac{1}{2}.$$

- If Firms are **patient enough** ($\delta \geq 1/2$) collusion can be sustained, and the **Bertrand paradox** solved

Outline

Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: quantity of a *homogeneous* good to be produced (infinite is not included in the production interval)

$$q_i = q_j = [0, \infty).$$

- Inverse market demand is $p(q) = a - Q$
- Payoffs: firm's profits

$$\pi_i = [a - Q - c]q_i,$$

- **The Firms set quantities an infinite number of times**

Cooperation vs Deviation (1)

- All payoffs between 0 and monopoly profit are **feasible**
- If firms collude and set half the **monopoly quantity**, price is **monopoly price** and each gets **half the monopoly profit**
- Recall that

$$q_i^{*c} = q_j^{*c} = \frac{a-c}{3} > \frac{a-c}{4} = q^m \text{ and}$$

$$\pi_i^{*c} = \pi_j^{*c} = \frac{(a-c)^2}{9} < \frac{(a-c)^2}{8} = \frac{\pi^m}{2}.$$

- The two Firms play the **trigger strategy**:
  "Produce half the monopoly quantity at stage t as long as the other Firm produced half the monopoly quantity at stage $t-1$. Otherwise, produce the Cournot quantity."

Cooperation vs Deviation - Deviation Quantity (2)

- **Deviation profits** in repeated Cournot are obtained by replacing half the monopoly quantity in the **maximisation problem** for firm *i*:

$$\max_{q_i}[a - q_i - \frac{a-c}{4} - c]q_i.$$

- Take the first derivative and equate it to zero:

$$\frac{\delta\pi_i(q_i, \frac{a-c}{4})}{\delta q_i} = a - 2q_i - \frac{a-c}{4} - c$$

$$8q_i = 4a - a + c - 4c = 0 \implies q_i^{dev} = \frac{3}{8}(a-c)$$

Cooperation vs Deviation - Deviation Profits (3)

- To find deviation profits for Firm i, plug q_i^{dev} in the profit equation:

$$\pi_i^{dev} = [a - \frac{3}{8}(a-c) - \frac{a-c}{4} - c]\frac{3}{8}(a-c) =$$

$$= \left[\frac{8a - 8c - 3a + 3c - 2a + 2c}{8}\right]\frac{3}{8}(a-c) =$$

$$= \frac{3}{8}(a-c)\frac{3}{8}(a-c) = \frac{9(a-c)^2}{64}.$$

- Notice that $\pi_i^{dev} > \pi^m/2$

Cooperation vs Deviation - Cooperative Outcome (3)

- The cooperative outcome can be achieved if

$$PV^{coop} \geq PV^{dev},$$

$$\frac{(a-c)^2}{8}+\delta\frac{(a-c)^2}{8}+\delta^2\frac{(a-c)^2}{8}+\ldots \geq \frac{9(a-c)^2}{64}+\delta\frac{(a-c)^2}{9}+\delta^2\frac{(a-c)^2}{9}+\ldots,$$

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8} \geq \frac{9(a-c)^2}{64}+\frac{\delta}{1-\delta}\frac{(a-c)^2}{9},$$

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{9}{64}+\frac{\delta}{1-\delta}\frac{1}{9} \implies \frac{1}{1-\delta}\frac{64}{8} \geq 9+\frac{\delta}{1-\delta}\frac{64}{9},$$

$$8 \geq 9-9\delta+\frac{64}{9}\delta \implies \delta(9-\frac{64}{9}) \geq 1,$$

$$\delta\left(\frac{81-64}{9}\right) \geq 1 \implies \delta \geq \frac{9}{17}.$$

Cooperation vs Deviation - Comments (4)

- The cooperative outcome can be achieved if $\delta \geq 9/17$
- Notice that the discount factor must be **larger in Cournot** than in Bertrand. This occurs as retaliation after deviation is less harsh in Cournot (profits are positive)
- **Extension**:
  - In case of different marginal costs, required δ would be higher for the **more efficient firm**, as deviation is more profitable
  - If deviation is observed only after **two (or more) periods** (this applies also to Bertrand), required δ is higher