Games, Information and Contract Theory
Problem Set 3

Instructors

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This problem set will be solved in class on Monday, October 7, 2019.

Exercise 1. (Formula 1)

Two drivers, *Al* and *Ham*, participate in a Formula 1 race. The game is as follows: in the first stage, *Al* chooses to sabotage (S) or not sabotage (NS) *Ham's* car. If this happens, *Ham's* car breaks down right after ignition, and the game ends. In the second and third stages (if no sabotage occurred), *Al* and *Ham* sequentially choose the type of tyres, *Rain* (R) or *Dry* (D), to be mounted on their cars. *Ham* does not observe the tyres chosen by *Al* in the second stage. Payoffs are determined as follows:

- (4,0) if *Al* sabotages *Ham's* car;
- (1,2) if *Al* does not sabotage, and both *Al* and *Ham* mount *Rain* tyres;
- (2,3) if *Al* does not sabotage and mounts *Rain* tyres, while *Ham* mounts *Dry* tyres;
- (5,4) if *Al* does not sabotage and mounts *Dry* tyres, while *Ham* mounts *Rain* tyres;
- (0,5) if *Al* does not sabotage, and both *Al* and *Ham* mount *Dry* tyres.

Answer the following questions and explain your answers in detail.

1. Write the game as an *extensive-form* game (*Hint: draw the game tree with payoffs first. Then write separately the set of players $N = \{\}$, the set of strategies available to each player $S_i = \{\}$ (remember the definition of strategy in dynamic games), the set of decision nodes $X_i = \{\}$ and of information sets $I_i = \{\}$ for each player, the root of the game $r = \{\}$, and the terminal nodes $TN = \{\}$*).

2. Identify all the *subgames* which compose this game.

3. Is there any *strictly dominant strategy* for *Al*? and for Ham?

4. Use *backwards-induction* to find the *subgame-perfect NE* of this game. Verify that this is an equilibrium of each *subgame*. (*Hint*: Remember that the subgame starting in stage 2 can be solved as a *static game of complete information*, as *Ham* does not observe *Al's* choice. As a consequence, write the latter as a normal-form game, and find its *NE*. Then proceed backwards.)

5. Suppose now that the last payoff is modified as follows: (0,3) if *Al* does not sabotage, and both *Al* and *Ham* mount *Dry* tyres. Does this affect the *NE* in the subgame starting in stage 2? Does this affect the *subgame-perfect NE* of this game?

Exercise 2. (Collusion in Cournot)

Consider a *Cournot game* in which two firms 1 and 2 produce a *homogeneous good* and interact an *infinite number of times*. Both firms have a common *discount factor* δ, with $0 < \delta < 1$, which is a measure of their *patience* concerning future profits. In each of the (infinite and identical) stage games, the two firms simultaneously set the quantity of the goods they produce $(q_1; q_2)$. The *marginal cost* of production is $c = 3$. Inverse market demand in each stage is given by

$$P(Q) = 9 - Q,$$

where $Q = q_1 + q_2$.

Answer the following questions and explain your answers in detail.

1. Find the *Nash equilibrium* quantity and profits of the *stage game*.

   Suppose that both firms play the following *trigger strategy*:

   "*In stage t, set* $q_1 = q^m/2$ *if firm* 2 *set* $q_2 = q^m/2$ *in all the stages before the current one; otherwise, set the Cournot quantity forever*"

2. What are the maximum profits that can be achieved by the two firms if they cooperate?

3. What is the q^{DEV}, i.e. the quantity produced by firm 1 if it deviates when the other plays the trigger strategy? Compute the deviation profits for firm 1.

4. Write the *present value* of the profits obtained by firm 1 if it *cooperates* and defects.

5. Find the conditions on δ such that the *trigger strategy* detailed above is a *SPNE* of the Cournot game.

Exercise 3. (Hiring Decisions)

There are a *Firm* (F) and a *Worker* (W). W can be of *high ability* ($T_w = high$), in which case he would like to *Work* when he is hired, or of *low ability* ($T_w = low$), in which case he would rather *Shirk*. F would want to *Hire* W if the latter is willing to *Work* and *Not Hire* him otherwise. W knows his ability level. F does not know the Worker's ability level, but it *believes* that his ability is high or low with probabilities of respectively $p(T_w = high) = \frac{2}{3}$ and $p(T_w = low) = \frac{1}{3}$. The payoffs (different for each type of W) are as follows:

	$Work$	$Shirk$
$Hire$	$1,2$	$0,1$
$Don't$	$0,0$	$0,0$

$T_w = High$

	$Work$	$Shirk$
$Hire$	$1,1$	$-1,2$
$Don't$	$0,0$	$0,0$

$T_w = Low$

Answer the following questions and explain your answers in detail.

1. Describe the game as a normal-form game (*hint: write separately the set of players $N = \{\}$, the set of actions available to each player $A_i = \{\}$, the type space T_i for each player, and the beliefs $p(t_w)$*).

2. Write the *strategy profile* for each player (*hint: recall that a strategy in Bayesian games is a function from the type space to the action space*).

3. Write the game as an extensive-form game (*hint: write the game tree. Remember to add a preliminary stage to the game, in which Nature draws t_w from T_w)*.

4. Find the *Bayesian NE* of this game (*hint: create a new payoff matrix in which you include the expected payoffs for F and the payoffs for W. Then look for best-responses for EACH type of player*).