Practice session 1

Game Theory - MSc EEBL

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September 19, 2024

Exercise 1. *Starters*

Consider the static game of complete information described by the following payoff matrix:

		Player 2 L	Player 2 C	Player 2 R
Player 1	T	(2,0)	(1,1)	(4,2)
Player 1	M	(3,4)	(1,2)	(2,3)
Player 1	B	(1,3)	(0,2)	(3,0)

1. Write the game as a normal-form game. That is, define the set of players N, the set of strategies S_i for each player $i \in N$, and the payoffs associated to each outcome of the game for each player $u_i(s)$ where $s = (s_1, \ldots, s_n) \in \bigtimes_{i \in N} S_i$.

2. Apply the concept of *Iterated Elimination of Strictly Dominated Strategies (IESDS)*, and write the strategies that survive this process (*Hint: Start with Player 1, then move to Player 2. Iterate*).

3. Find the *pure-strategy Nash equilibria (NE)* of this game.

4. Compare your findings in questions 2 and 3. Explain the relationship between *IESDS* and *NE*.

Answer of Exercise 1.

1. To write the game as a *normal-form* game, we detail the set of **Players**, **Strategies**, and players' **Payoffs** from each combination of strategies.

   - Players: $N = \{\text{Player 1}, \text{Player 2}\}$.
   - Strategy sets: $S_1 = \{T, M, B\}$ and $S_2 = \{L, C, R\}$.
   - Payoffs:

$$u_1(T,L) = 2, u_1(T,C) = 1, u_1(T,R) = 4, u_1(M,L) = 3, u_1(M,C) = 1,$$

$$u_1(M,R) = 2, u_1(B,L) = 1, u_1(B,C) = 0, u_1(B,R) = 3.$$

$$u_2(T,L) = 0, u_2(T,C) = 1, u_2(T,R) = 2, u_2(M,L) = 4, u_2(M,C) = 2,$$

$$u_2(M,R) = 3, u_2(B,L) = 3, u_2(B,C) = 2, u_2(B,R) = 0.$$

2. In order to carry out *IESDS*, we check for **strictly dominated pure strategies** for Players 1 and 2. Notice that at the present stage there are no strictly dominated pure strategies for Player 2 (check this):

   - For Player 1, B is strictly dominated by T (2>1, 1>0, 4>3) in the sense that T is better than B in terms of payoffs regardless of what Player 2 does. Since Player 1 is rational, she will never play B. Moreover, Player 2 knows that 1 is rational and will thus never play B. We can erase the corresponding row.

Player 2

		L	C	R
Player 1	T	(2,0)	(1,1)	(4,2)
	M	(3,4)	(1,2)	(2,3)

   - Notice that now C is strictly dominated by R for Player 2 (2>1, 3>2). As Player 2 is rational she will never play C. Moreover, Player 1 knows that 2 is rational and will never play C. We can erase the corresponding column.

Player 2

		L	R
Player 1	T	(2,0)	(4,2)
	M	(3,4)	(2,3)

   - Notice that there are no more strictly dominated pure strategies for the two players: for 1, T is better than M when 2 plays R but M is better than T when 2 plays L; for 2, L is better than R when 2 plays M but R is better than T when 2 plays T. We cannot cancel out any other pure strategy and, as a consequence, we need to resort to a stronger solution concept.

3. To find the **pure-strategy NE** of this game, we underline each player's **best responses** to the strategies of the other.

Player 2

		L	C	R
	T	(2,0)	$(\underline{1},1)$	$(\underline{4},\underline{2})$
Player 1	M	$(\underline{3},\underline{4})$	$(\underline{1},2)$	(2,3)
	B	$(1,\underline{3})$	(0,2)	(3,0)

There are two **pure-strategy NE** in this game, i.e. $\{M, L\}$ and $\{T, R\}$.

4. There are more strategy profiles surviving *IESDS* than *NE*. In particular, $\{M, L\}$, $\{M, R\}$, $\{T, L\}$, and $\{T, R\}$ survive *IESDS* but only $\{M, L\}$ and $\{T, R\}$ are pure-strategy *NE* of the game. As expected, *NE* survive *IESDS*, but not all surviving strategy profiles are pure-strategy *NE*.

Exercise 2. *Coordination, conflict and efficiency*

1. Consider the game of Exercise 1.

   (a) Find the Pareto efficient outcome(s).

   (b) Compare with the Nash equilibria obtained in question 3 (Exercise 1), what can you say?

2. Consider now the following game. Two friends, A and B, want to have a drink and there are $n \in \mathbb{N}$ bars open denoted by B_1, B_2, ..., B_n. Unfortunately, A's cellphone is out of battery and they have not decided in which bar they wanted to go. The payoff matrix is as follows.

Player 2

		B_1	B_2	B_3	$\dots$	B_n
	B_1	$(1,1)$	$(0,0)$	$(0,0)$	$\dots$	$(0,0)$
	B_2	$(0,0)$	$(1,1)$	$(0,0)$	$\dots$	$(0,0)$
Player 1	B_3	$(0,0)$	$(0,0)$	$(1,1)$	$\dots$	$(0,0)$
	$\vdots$	$(0,0)$	$(0,0)$	$(0,0)$	$\dots$	$(0,0)$
	B_n	$(0,0)$	$(0,0)$	$(0,0)$	$\dots$	$(1,1)$

(a) What can you say about the nature of the game? Is there any problem of conflicting interests? Of coordination?

(b) Find the pure-strategy Nash equilibria.

(c) Find the Pareto efficient outcome(s). Compare with the NE.

(d) Assume A finds a plug to load their battery and can communicate with B, what could the two friends do?

3. Consider now the following game. Two coworkers, C_1 and C_2 are annoyed by a flickering light in their office. Each of them can report it by sending an email to the responsible person at a small, but positive individual cost $c \in (0,1)$. The payoff matrix is as follows.

		C_2: Report	C_2: Say nothing
C_1	Report	$(1-c, 1-c)$	$(1-c, 1)$
	Say nothing	$(1, 1-c)$	$(0,0)$

(a) Characterize the pure-strategy Nash equilibria for all possible values of $c \in (0,1)$.

(b) Find the Pareto efficient outcome(s). Compare with the NE when $c \in (0,1)$.

(c) What can you say about conflicting interests? And coordination? What is the importance of the individual cost, c, here? Comment.

4. Consider now the following game. Two pharmaceutical laboratories, P_1 and P_2, want to develop a new drug. They can either team up and aim at developing a new generation drug or each can work on its own to develop a less ambitious drug. However, they cannot work both on the common project and on the solo project so that if P_i works on the common project while P_j ($j \neq i$) works on the solo project, the common project will fail. The payoff matrix is as follows.

		P_2: Common	P_2: Solo
P_1	Common	$(2,2)$	$(0,1)$
	Solo	$(1,0)$	$(1,1)$

(a) Find the pure-strategy Nash equilibria.

(b) Find the Pareto efficient outcome(s). Compare with the NE.

(c) What can you say about conflicting interests? And coordination?

5. Consider now the following game. Consider two suspected individuals, A and B, which are interrogated, one by one, by a police officer. Each can decide to either stay quiet or to provide evidence to the police officer. If both individuals stay quiet, they only have to pay for a small fine but cannot be further convicted due to the lack of evidence. If one provides evidence while the other stays quiet, the talkative one gets a sentence reduction while the quiet one is charged with the criminal offense alone. If both give evidence, they are both charged with the criminal offense and share its responsibility. The payoff matrix is as follows.

		B: Quiet	B: Talk
A	Quiet	$(1,1)$	$(-1,2)$
	Talk	$(2,-1)$	$(0,0)$

   (a) Find the pure-strategy Nash equilibria.

   (b) Find the Pareto efficient outcome(s). Compare with the NE.

   (c) If the two individuals were able to communicate prior to the questioning by the police officer would they be able to reach the situation $\{\text{Quiet}, \text{Quiet}\}$? What if one could commit not to talk?

Answer of Exercise 2.

1. (a) To find the Pareto optimal outcomes we have to look at each payoff for each combination of strategies and investigates whether there exist another combination of strategies that gives a strictly higher payoff to Player i and does not make Player $j \neq i$ worse-off (Player j can be either indifferent or better-off with the new combination of strategies).

   A good starting point consists in identifying a combination of strategies that gives quite high payoffs to both players and compare it with other combinations to eliminate them.

   Here, a good candidate is the combination $\{M, L\}$ which gives 3 to Player 1 and 4 to Player 2. It is then easy to see that both players would (strictly or weakly) prefer $\{M, L\}$ to any element in $\{\{T, L\}, \{B, L\}, \{T, C\}, \{M, C\}, \{M, R\}, \{B, C\}, \{B, R\}\}$.

   Therefore, the only remaining combination of strategies is $\{T, R\}$, which gives 4 to Player 1 and 2 to Player 2. Notice that if we start at $\{M, L\}$, moving to $\{T, R\}$ is beneficial for Player 1 but is detrimental to Player 2. Therefore, starting from $\{M, L\}$, there is no Pareto improvement so that we can say that $\{M, L\}$ is a Pareto optimal.

Similarly, starting from $\{T, R\}$ and moving to $\{M, L\}$ would be beneficial to Player 2 but detrimental to Player 1. Thus, $\{T, R\}$ is also a Pareto optimal.

(b) In that case, the Nash equilibria coincide with the Pareto optimal outcomes. Be careful, however, this is purely incidental here. In general, there is no reason for equilibrium strategies to coincide with the Pareto optimal outcomes. This is even what motivates economists to work on such topics as players generally fail to reach an efficient outcome if they are left to themselves!

2. (a) This game is a **pure coordination problem**. The interests of both players are perfectly aligned, i.e., they both have an interest to meet in the same bar it is just that they do not know which one.

(b) The pure-strategy equilibria are simply the collection of $\{B_k, B_k\}$ for all $k = 1, \dots, n$, that is, when the two friends choose to meet in the exact same bar (the diagonal in the payoff matrix). The problem, however, is that the concept of Nash equilibrium does not help us to go further and say what the two friends will actually do and how they are going to coordinate. We only know what is an equilibrium and what is not.

(c) As in the previous question, let us choose a combination of strategies that gives quit high payoffs to both players, say $\{B_1, B_1\}$ that gives 1 to each player. It is straightforward to see that any combination $\{B_i, B_j\}$ with $i \neq j$ can be eliminated as they give 0 to each player. At the same time any combination $\{B_k, B_k\}$ for all $k = 1, \dots, n$ gives 1 to both players such that they are all Pareto efficient.
Once again here, Pareto efficiency coincide with the Nash equilibria.

(d) If we assume that players could communicate before playing, then it appears that they could be able to coordinate on any NE. For instance, player A could tell player B that they will meet at B_4 and they will both choose to actually go to B_4 as it is in their common interest.
Notice that this occurs because we are in a pure coordination game with no conflicting interests. We will see later than even when players can communicate and that it seems that it could improve their situation, they may fail to reach a better outcome.

3. (a) There are two pure-strategy Nash equilibria here, $\{R, SN\}$ and $\{SN, R\}$ where R and SN stands for "Report" and "Say Nothing", respectively. In each equilibrium, one of the player "volunteer" to do the reporting and the other one free rides on the volunteer. Still, the task is done in the end.

(b) Clearly $\{R, R\}$ cannot be Pareto efficient as it is better that one agent free rides while the other does the report than sending two reports. Same for $\{SN, SN\}$, it is better that one the player volunteers for sending the report. However, starting from either $\{R, SN\}$ or $\{SN, R\}$, there is no way to increase the payoff of one the agent without harming the other one. Hence, both $\{R, SN\}$ and $\{SN, R\}$ are Pareto efficient. Again, Pareto efficient outcomes coincide with Nash Equilibria.

(c) In this game, there is a coordination problem as players should not both write the report or both choose to free ride. At the same time, there is also a conflict of interests, as each player prefers not to write the report and let the other write it instead. Yet, the conflict of interest is not total as they also have a common interest in stopping the light flickering.

4. (a) There are two pure-strategy Nash equilibria here, $\{C, C\}$ and $\{S, S\}$. See that one seems much better than the other.

   (b) There is only one Pareto efficient outcome here and it is $\{C, C\}$. It coincides with one the Nash equilibrium but it means that the other Nash equilibrium $\{S, S\}$ is not Pareto efficient. In other words, it means that at equilibrium we could have an inefficient outcome. Hence, this shows that even rational players, maximizing their payoffs, may fail to achieve the efficient outcome and be trapped in a bad equilibrium.

   (c) In this game, there is still a coordination problem as we have seen that players could rationally end up in the "bad" equilibrium, namely $\{S, S\}$ whereas it would be in their common interest to coordinate on the "good" equilibrium, namely $\{C, C\}$. However, there is also conflicting interests as each player is tempted to play solo to avoid loosing everything if the other one does not choose to work on the common project.

   If, for instance, P_1 was able to show proof that he/she is working on the common project, then P_2 would surely choose to also work on the common project and the "good" equilibrium would prevail.

5. (a) There is a unique Nash equilibrium which is $\{T, T\}$ that is both players choose to talk and provide evidence to the police officer. This seems to be a very "bad" outcome.

   (b) For sure, $\{Q, Q\}$ is Pareto efficient as moving to any other combinations of strategies would decrease the payoff of at least one player. But do not forget that $\{T, Q\}$ and $\{Q, T\}$ are also Pareto efficient. The only combination which is not Pareto efficient is $\{T, T\}$. So now, we have that the Nash equilibrium is the only combination of strategies that is not Pareto efficient!

   (c) Communication and commitment seemed to be good allies to "restore" some efficiency as we have seen previously. Let us see how they perform in this game.

   First, consider only the possibility of communication prior to the questioning. Maybe the two individuals will say to each other "let us stay quiet and avoid too harsh sentences" (i.e. trying to reach $\{Q, Q\}$). But once they will be questioned by the police officer, each individual finds it better to talk rather than to stay quiet. Communication prior to the questioning does not help here (not to mention if one individual says to the other that he/she will talk the other will surely talk as well).

   So now let us assume that not only they can communicate but one can also commit to a strategy. For instance, assume that A can commit to stay quiet. Same thing happens, B will surely decide to talk to get the sentence reduction so that neither communication nor commitment on one side are sufficient to restore efficiency.

Here the conflict of interest is so strong that players are driven to the worst possible scenario.

Exercise 3. *Study together*

Ann and Paul have to study for their Game Theory exam. They can decide to study at their own home or at the university library. If they both remain at home, Ann's payoff is 2 and Paul's payoff is 0; if they both study at the university, they study together and Ann's payoff is equal to Paul's payoff it is denoted by $x \in \mathbb{R}_+$; if only Ann goes to the library, Ann's payoff is 1 and Paul's payoff is -1; otherwise, if only Paul goes to the library, Ann's payoff is 2 and Paul's payoff is 1.

		Paul	
		Home	Library
Ann	Home	(,)	(,)
	Library	(,)	(x,x)

1. Fill the payoff matrix using the available information.
2. Write the game as a normal-form game.
3. Can "study at the library" be a dominant strategy for Ann? Can "study at home" be a dominant strategy for Ann? And for Paul? Explain (*Hint: this depends on the values taken by x*).
4. Characterize the Nash equilibria depending on the value of $x \in \mathbb{R}_+$. Is is possible to have "both go to the library" as a Nash Equilibrium?

Answer of Exercise 3.

1. Using the information about the payoffs, the matrix writes:

		Paul	
		Home	Library
Ann	Home	(2,0)	(2,1)
	Library	(1,-1)	(x,x)

2. The normal-form writes:
   - Players: $N = \{\text{Ann}, \text{Paul}\}$.

- Strategies: $S_A = \{H, L\}$ and $S_P = \{H, L\}$.
- Payoffs:

$$u_A(H,H) = 2, u_A(H,L) = 2, u_A(L,H) = 1, u_A(L,L) = x.$$
$$u_P(H,H) = 0, u_P(H,L) = 1, u_P(L,H) = -1, u_P(L,L) = x.$$

3. - L cannot be a dominant strategy for Ann for any x (either positive or negative). This occurs as H is better than L for Ann when Paul plays H ($2 > 1$).
   - L can be a weakly dominant strategy for Paul. This requires $x \geqslant -1$. L is strictly dominant if $x > -1$ (L is better when Ann plays H).

4. For $\{L, L\}$ to be a **pure-strategy NE**, we need L to be a **best response** L for both players. For Paul this occurs when $x \geqslant -1$, while for Ann when $x \geqslant 2$. When $x > 2$ then $\{L, L\}$ is the only pure-strategy NE. If $x = 2$ then both $\{L, L\}$ and $\{H, L\}$ are NE.

Reminder.

- **Normal-Form Representation of Games.**

  A game in normal (or strategic) form has three elements:

  1. A set of players $N = \{1, \ldots, n\}$ which we consider to be a finite set.
  2. Pure-strategy space S_i for each player $i \in N$. We note $S = \times_i S_i$ the strategy space and s_i an element of the set S_i.
  3. Payoff functions u_i for each player $i \in N$ where $u_i : S \to R$.

- **Dominant and Dominated Strategies.**

  Definition: The strategy $s_i \in S_i$ **strictly** dominates the strategy s_i' if:

  $$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}), \text{ for all } s_{-i} \in S_{-i}.$$

  Definition: The strategy s_i is **strictly** dominant if it **strictly** dominates s_i' for all $s_i' \neq s_i$.

  Definition: The strategy $s_i \in S_i$ **weakly** dominates the strategy s_i' if:

  $$\forall\, s_{-i} \in S_{-i} \quad u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}) \quad \text{(with at least one } s_{-i} \text{ that gives a strict inequality)}$$

- **Nash Equilibrium.**

  The strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a **Nash Equilibrium** if for all $i = 1, \ldots, n$ and all $s_i \in S_i$ we have:

  $$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*).$$

- **Best response.**

  Definition: $s_i \in S_i$ is a **best response** to $s_{-i} \in S_{-i}$ if:

  $$u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}), \text{ for all } s_i' \in S_i.$$

  Definition (more technical): the **best response correspondence** of player i is a correspondence $BR_i : S_{-i} \rightrightarrows S_i$ given by:

  $$BR_i(S_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

  Note: we use the term correspondence and not function because $BR_i(\cdot)$ is not necessarily single-valued. A function $f : X \to Y$ maps every element $x \in X$ to *one and only one point* $y \in Y$. A correspondence $g : X \rightrightarrows Y$, however, maps every element $x \in X$ to the *power set* of Y, namely 2^Y (*i.e.* the set of all subsets of Y).

Additional material. *Simple problems*

In this additional exercise, you can simply train yourself to find all the pure-strategy Nash equilibria in different games.

1. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(2,0)	(1,1)
	D	(3,4)	(1,2)

2. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(-1,3)	(-2,-1)
	D	(2,1)	(4,-3)

3. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(-2,5)	(-3,-1)
	D	(2,0)	(-2,3)

4. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(4,1)	(-7,-3)
	D	(8,-3)	(-2,5)

5. Find all pure-strategy Nash equilibria of the following game.

			Player 2	
		L	C	R
	U	(2,0)	(1,3)	(2,4)
Player 1	M	(3,4)	(1,2)	(0,3)
	D	(2,1)	(5,2)	(2,3)

6. Find all pure-strategy Nash equilibria of the following game.

		Player 2			
		A	B	C	D
	U	(3,3)	(1,0)	(0,3)	(2,2)
Player 1	M	(3,3)	(0,0)	(3,2)	(0,2)
	D	(4,1)	(2,2)	(2,0)	(3,1)

Answer of Exercise 3.

1. Solution: $\{U, R\}$ and $\{D, L\}$.
2. Solution: $\{D, L\}$.
3. Solution: $\{D, R\}$.
4. Solution: $\{D, R\}$.
5. Solution: $\{M, L\}$, $\{U, R\}$ and $\{D, R\}$.
6. Solution: $\{D, B\}$.