Game Theory: Dynamic Games of Complete Information

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Dynamic games

So far we have investigated **static games.**

- Players had to **simultaneously** choose an action and then the **game ended.**

In **dynamic games**, there will be **more than one period** of play.

- Players may have to choose more than one action.
- Players might **observe** other players' past choice of action.

Table of Contents

Dynamic games: A first example

One classical example of dynamic games is an **entry game.**

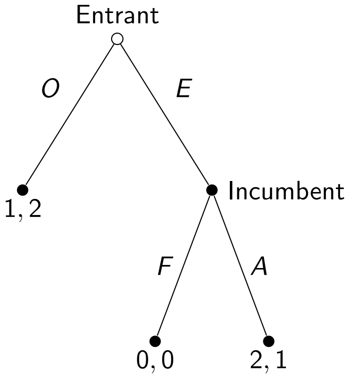
Consider two firms, an **incumbent** and a **potential entrant**.

The potential entrant **plays first** and decide whether to **enter** (E) the market or to **stay out** of it (O).

If the entrant decided to enter, the incumbent can choose either to **fight** them F or to **accommodate** A.

Dynamic games: A first example

A convenient way to describe those games is to draw **a tree.**



The first number is the entrant's payoff.

Dynamic games: A first example

What is peculiar about this game?

First, the incumbent can play only if the entrant has decided to enter.

Second, when the incumbent plays, they know that the Entrant has chosen to enter.

- The incumbent really **observes** the action chosen by the entrant.

Third, we account for players' payoffs only **at the end of the game.**

A first example: Game tree

This game seems different from the ones we have seen previously.

We will have to find a way to **describe those games:** Players, Rules, Payoffs.

And a way to solve them: **A solution concept.**

We can start by trying to apply the tools we have already introduced.

- Namely, the **normal-form** representation and **Nash equilibrium.**

A first example: Normal-form?

Normal-form. Players $N = \{\text{Entrant}, \text{Incumbent}\}$, $A_E = \{O, E\}$, $A_I = \{F, A\}$ and payoffs can be described using a payoff matrix:

Ent.\In.	F	A
O	$1, 2$	$1, 2$
E	$0, 0$	$2, 1$

The first problem we encounter with this description is that we do not really understand what it means to play F and A when the entrant stays out.

A first example: Normal-form?

So, do we have to simply **ignore** the existence of F and A when the entrant chooses O?

But when you play such a game, it is not because you are not able to play that you cannot still **make a plan** about what you would have played in that case.

This is where the notions of **actions** and **strategies** start to become really different objects.

A first example: Actions and strategies

Actions should be considered as the *physical* choices available to agent.

- It describes what they are able to do at each period or after moves of other players.

Strategies must be thought more like *plans* that players formulate in their mind.

- If X happened I would choose to do Y.
- Does not necessarily occur at any moment.
- But the plan is there if the situation occurs.

A first example: Nash equilibrium?

Hence, if we want to think about Nash equilibrium, we have to think in terms of strategies and not in terms of actions.

Ent.\In.	F	A
O	$1, 2$	$1, 2$
E	$0, 0$	$2, 1$

In our example, when the entrant thinks about staying out, they must try to anticipate what the incumbent would do if instead they decided to enter.

A first example: Nash equilibrium?

Nash equilibrium. If we apply our Nash equilibrium concept to this game we obtain a solution.

Ent.\In.	F	A
O	$\underline{1}, \underline{2}$	$1, \underline{2}$
E	$0, 0$	$\underline{2}, \underline{1}$

We obtain two Nash Equilibria: (O, F) and (E, A).

A first example: Nash equilibrium?

Ent.\In.	F	A
O	$\underline{1}, \underline{2}$	$1, \underline{2}$
E	$0, 0$	$\underline{2}, \underline{1}$

The Nash Equilibrium (E, A) is easy to interpret.

- The Entrant enters and the Incumbent Accommodates.

A first example: Nash equilibrium?

Ent.\In.	F	A
O	$\underline{1}, \underline{2}$	$1, \underline{2}$
E	$0, 0$	$\underline{2}, \underline{1}$

The Nash Equilibrium (O, F) is a bit more subtle to interpret:

- The Entrant stays out and the Incumbent had planned to fight in case the Entrant entered.

The Incumbent's strategy is a **threat**.

A first example: Nash equilibrium?

Ent.\In.	F	A
O	$\underline{1}, \underline{2}$	$1, \underline{2}$
E	$0, 0$	$\underline{2}, \underline{1}$

Do you find (O, F) *satisfying* as a solution to this game?

We will see why the Nash equilibrium is not our first choice as a solution concept for dynamic games.

Table of Contents

Extensive-form representation

Dynamic games can be represented with what we call the **extensive-form representation**.

Like the normal form, the extensive form describes a wide range of games.

It contains *more information* than the normal form because it specifies also when players can play and what can they do in each of their move.

Extensive-form representation

The extensive form of a game specifies.

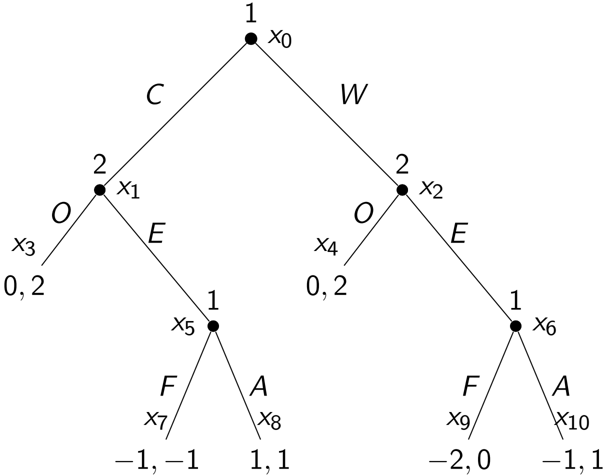
- Players.
- When each player can play.
- What each player can play when it is their turn.
- The payoff of each player for every possible **terminal history** of the game.

Extensive-form representation

More formally, the extensive form of a game specifies.

- Players: $N = \{1, \ldots, n\}$.
- Nodes: A set $X = \{x_1, \ldots, x_p\}$.
  - A **Root**, or starting point: $r \in X$.
  - **Decision nodes** for each player: $X_i \subseteq X$.
  - **Terminal nodes**: $T \subseteq X$.
- Actions available at each decision node: A_{x_i} for $x_i \in X_i$.
- **Payoffs** at each terminal nodes: $u_i : T \to \mathbb{R}$.

Extensive form: An Example



Extensive-form: An example

- Players: $N = \{1, 2\}$.
- Nodes: $X = \{x_0, \ldots, x_{10}\}$.
  - A **Root**, or starting point: $r = x_0$.
  - **Decision nodes**: $X_1 = \{x_0, x_5, x_6\}$, $X_2 = \{x_1, x_2\}$.
  - **Terminal nodes**: $T = \{x_3, x_4, x_7, x_8, x_9, x_{10}\}$.
- Action spaces: $A_{x_0} = \{C, W\}$, $A_{x_1} = A_{x_2} = \{O, E\}$, $A_{x_5} = A_{x_6} = \{F, A\}$.
- **Payoffs** at each terminal nodes:
  - For instance $u_1(x_7) = -1$ and $u_2(x_9) = 0$.

But we can use also the usual notation $u_1(C, E, A) = 1$.

Table of Contents

Strategies and actions

In order to define possible solution concepts we need to precisely define what is a **strategy in an extensive-form game.**

As said previously, a strategy is now a **different object** from an action.

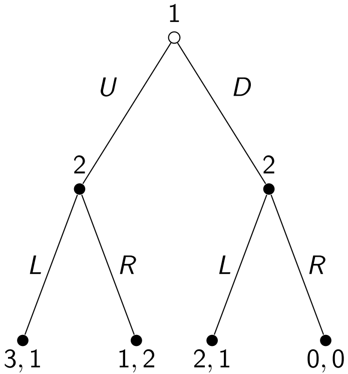
The set of actions of player i is simply a **description of what is available to this player** at each node they are asked to play.

A strategy of player i is a **complete plan of actions for all possible contingency.**

- It describes what player i plans to play at every node they are asked to play.
- Be careful: The strategy also assigns an action to nodes that "should not be reached".

Strategies: An example

Consider the following game.



Strategies: An example

In this game, player 1 has to play first and must choose between U and D.

- For them, an action and a strategy **is the same.**
- We write $S_1 = \{U, D\}$.

S_1 describes the complete plan of actions of P1 because they have to play only in one node.

It is similar to what we have seen in static games.

Strategies: An example

Player 2 has to choose between L and R but in **two possible scenarios:** After P1 played U or after P1 played D.

- For them, an action and a strategy **is different.**
- We write $S_2 = \{LL, LR, RL, RR\}$.

S_2 describes the **complete plan of actions** of P2 with the convention that:

- The first letter corresponds to P2's plan after U.
- The second letter corresponds to P2's plan after D.

Strategies: An example

Hence, LL corresponds to the strategy

- Play L after U.
- Play L after D.

And RL corresponds to the strategy

- Play R after U.
- Play L after D.

And so on.

Strategies: Why?

Why should we define a complete plan of actions?

- Couldn't we just "wait" to see what P1 does and define P2's action **only for this case?**

Assume we do that.

- How could P1 even decide what to do if they cannot **anticipate** what P2 would play in each scenario?

Strategies: Why?

When you play **Chess** and have to decide whether to move the king to E4 or D5.

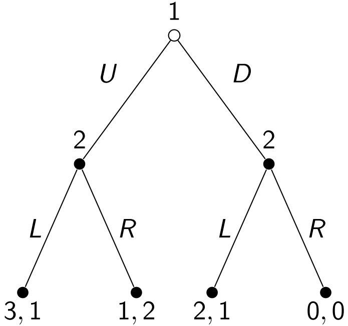
- You end up moving the king **only to one** of those location.
- But you had to think about the **consequences of both moves** in terms of subsequent actions of the other player.
- So you have in mind that the other player would have played different actions in each case.

Table of Contents

Nash equilibrium

Now that we have defined what is a strategy in a dynamic game let us try to apply the **Nash equilibrium concept to them.**

Let us consider this game once again:



Nash equilibrium

We could represent this game in normal form with the associated payoff matrix.

1 \2	LL	LR	RL	RR
U	$3,1$	$3,1$	$1,2$	$1,2$
D	$2,1$	$0,0$	$2,1$	$0,0$

It important to understand that for P2 we must take into account strategies and not actions.

- That way, when P1 decides between U and D against LR, for instance.
- P1 actually compares the history U followed by L to the history D followed by R.

Nash equilibrium

Underlining **best responses** we obtain:

$1 \backslash 2$	LL	LR	RL	RR
U	$\underline{3}, 1$	$\underline{3}, 1$	$1, \underline{2}$	$\underline{1}, \underline{2}$
D	$2, \underline{1}$	$0, 0$	$\underline{2}, \underline{1}$	$0, 0$

Be careful when looking for P2's BR:

- Both *RL* and *RR* "are" best response to *U*.
- It is solely because when considering P2's BR against *U* we should only consider the first letter in their strategies,
- i.e., the letter that corresponds to P2's choice **after** *U*.

Nash equilibrium

Hence,

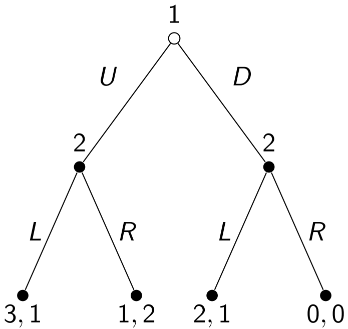
$1 \backslash 2$	LL	LR	RL	RR
U	$\underline{3}, 1$	$\underline{3}, 1$	$1, \underline{2}$	$\underline{1}, \underline{2}$
D	$2, \underline{1}$	$0, 0$	$\underline{2}, \underline{1}$	$0, 0$

tells us that (U, RR) and (D, RL) are **Nash equilibria** of this game.

- It is important to keep the full strategy of P2 (i.e. both letters) even if one seems useless.
- In (D, RL), knowing that P2 would have played R against U is as important as knowing that P2 plays L after D.

Nash equilibrium

Consider (U, RR) and (D, RL) in the game tree.



Could you confirm that indeed both are Nash equilibria in the sense that **no player is willing to deviate?**

Nash equilibrium

Let us start with (D, RL).

If P1 **believes that**

- P2 plays R after U.
- P2 plays L after D.

Then playing U yields 1 while playing D yields 2.

- P1 has **no incentive to deviate** and play U instead of D.

Nash equilibrium

Still considering (D, RL).

After observing D, P2

- obtains 1 by playing L.
- obtains 0 by playing R.

Then P2 has **no incentive to deviate** and play R instead of L after D.

Nash equilibrium

Then consider (U, RR).

If P1 **believes that**

- P2 plays R after U.
- P2 plays R after D.

Then playing U yields 1 while playing D yields 0.

- P1 has **no incentive to deviate** and play D instead of U.

Nash equilibrium

Still considering (U, RR).

After observing U, P2

- obtains 1 by playing L.
- obtains 2 by playing R.

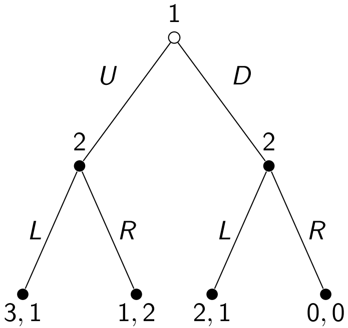
Then P2 has **no incentive to deviate** and play L instead of R after U.

Nash equilibrium: Satisfying?

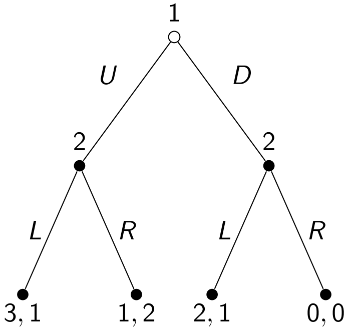
Are we good then?

- (U, RR) and (D, RL) are indeed Nash Equilibrium.

But let us have a closer look at (U, RR).



Nash equilibrium: Satisfying?



What does P2 prefers after D?

- L and not R.

Nash equilibrium: Satisfying?

Hence it seems *intuitive* that if D was played, then P2 would always play R and not L.

But we have found that (U, RR) was a Nash equilibrium.

- So it means that P2 **has formulated the plan** to play R after D.

Where is the *mistake* here?

There is no real *mistake* but it is the **weakness of the Nash equilibrium as a solution concept in dynamic games.**

Nash equilibrium: Satisfying?

When we have **considered deviations of P1**, we have used the **unilateral deviation** principle and considered the deviation of P1 against the fix strategy RR.

And thus we **assumed** that if P1 deviated to D, P2 **would stick to their plan** and play R after D.

- But intuition suggests us that in that case P2 would have then chosen L and not R.
- Because it is a dynamic game and P2 can now observe P1's choice before playing.

Nash equilibrium: Satisfying?

It seems that the strategy RR is not **sequentially rational.**

We like to call this strategy a **non-credible threat.**

Because it's a **threat** that makes P1 choose U instead of D, *fearing* that playing D would lead to R.

- But if P1 were actually to choose D then it would be rational for P2 to play L.
- Hence the initial threat is not credible.

Beyond Nash equilibrium

It is not that we got the wrong Nash equilibria.

It is simply that this was a **solution concept tailored to static games.**

We are now going to introduce a solution concept that takes this problem into account.

- Namely, **Subgame-perfect Nash equilibrium.**

Table of Contents

Subgame-perfect Nash equilibrium

The new solution concept we introduce now is the
Subgame-perfect Nash equilibrium.

The name obviously suggests that it is going to work **in a similar fashion as the Nash equilibrium.**

The new important word is: **Subgame.**

- Let us define what it means.

What are we trying to improve?

Recall that some solutions obtained with the Nash equilibrium concept **were not satisfactory.**

- Some players' strategy were **not sequentially rational.**

More precisely, this may happen to strategies **planned for histories not played at equilibrium.**

- That is, what player *i* had planned to do for some path of plays that should not occur if everyone were playing their equilibrium strategies.

What are we trying to improve?

The *problem* is the way the Nash equilibrium **deals with deviations.**

We cannot deal with the fact that if player i deviates then player j **might want to change their strategy after observing player i's action.**

- Because the Nash equilibrium allows only for **unilateral deviations** for the whole game.

We are not going to change the concept of **unilateral deviation** but we are going to *break* the game into **subgames** to account for **rational behavior at every stage.**

Subgames

The notion of **subgames** is simple.

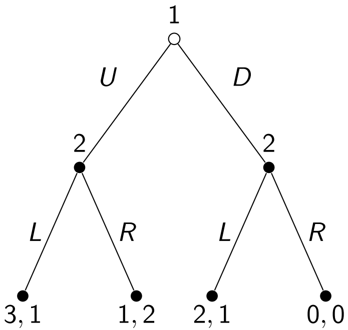
In a dynamic game, the game **does not necessarily end** after a player has taken an action.

If there are **remaining stages to be played**.

- We can see those remaining stages **as a game itself.**
- Namely, as **subgames** of the original game.

Subgames: An example

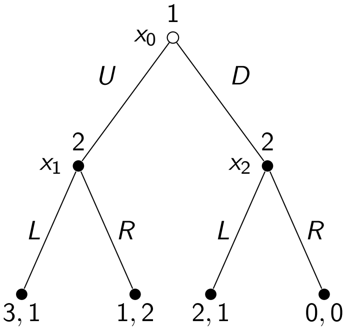
Consider the following game.



What could we call a **subgame** here?

Subgames: An example

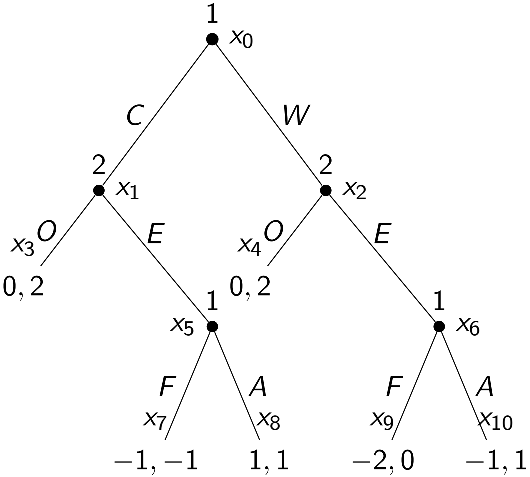
- At node x_1 starts a subgame in which only player 2 has to play.
- Same at node x_2.
- And at node x_0 starts a subgame which is the game itself.



Subgames: Another example

Subgames:

- The game itself at x_0.
- One at x_1.
- One at x_2.
- One at x_5.
- One at x_6.



Subgames: Definition

A subgame of an extensive-form game is therefore a game that **starts at a decision node of some player.**

All subgames of a given game can be identified by $\cup_{i \in N} X_i$.

- That is, **the set of all decision nodes of all players.**

Technically, we could say that there are also subgames starting at terminal nodes.

- But those subgames are irrelevant because they represents games in which players have no action at all.

Subgames of interest for us are those for which **at least one player still have an action to play.**

Subgame: Sequential rationality

We can somewhat think about all those subgames independently.

Because at the start of each subgame, **actions taken in the past cannot be changed** but are observed by players who still have to play.

- Future actions cannot affect past actions.
- Subgames can be seen as *independent* games themselves.

Hence, it seems reasonable to think that each of those subgames **should be be played rationally by players.**

- This is exactly the idea of Subgame-perfect Nash equilibrium.

Subgame-perfect Nash Equilibrium

Definition. A Nash equilibrium is **subgame-perfect** if the players' strategies constitute a Nash equilibrium **in every subgame.**

Remark: Notice that we still consider Nash equilibria but we **impose additional conditions** on them, namely, subgame perfection.

- We say that SPNE is **a refinement** of NE.

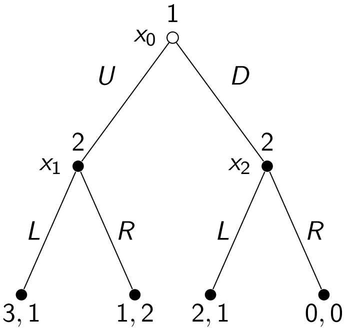
A SPNE is a NE, but a NE is **not necessarily** a SPNE.

- The set of SPNE is included in that of NE.

Subgame-perfect Nash Equilibrium: An example

In this game we know that **there are two NE** of the (sub)game starting at x_0.

They are (U, RR) and (D, RL).

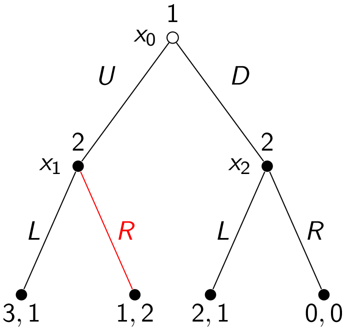


Subgame-perfect Nash Equilibrium: An example

There are **two other subgames,** at x_1 and at x_2.

For the one starting at x_1, only P2 has to take an action.

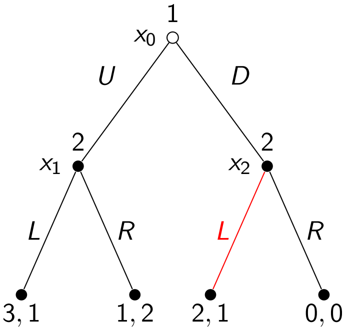
- Clearly P2 prefers R to L.
- P2 playing R is a **Nash equilibrium of this subgame.**



Subgame-perfect Nash Equilibrium: An example

For the one starting at x_2, only P2 has to take an action.

- P2 prefers L to R.
- P2 playing L is a **Nash equilibrium of this subgame.**



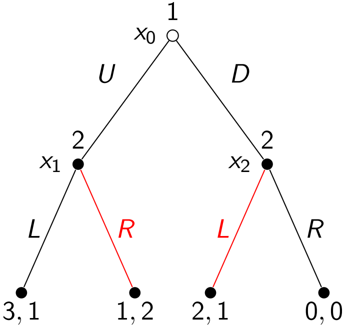
Subgame-perfect Nash Equilibrium: An example

Consider the NE of the game (D, RL).

Players' strategies **are a Nash equilibrium in every subgame.**

- (D, RL) of the one starting at x_0.
- R of the one starting at x_1.
- L of the one starting at x_2.

Hence (D, RL) is a **SPNE of the game.**



Subgame-perfect Nash Equilibrium: An example

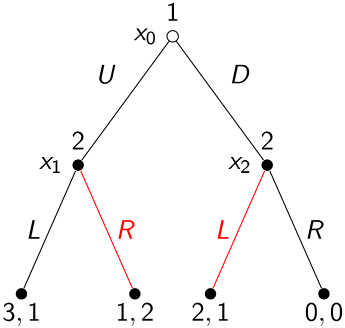
Consider now (U, RR).

As before,

- R is the NE of the subgame starting at x_1.
- L is the NE of the subgame starting at x_2.

It means that (U, RR) **does not satisfy the SPNE requirements.**

- Playing R in the subgame starting at x_2 **is not a NE of this subgame.**



SPNE: Noncredible threats

In the previous example, there is **only one of the two NE which is also a SPNE.**

The SPNE solution concept allows us to **eliminate** the NE in which P2 had a **noncredible threat.**

Imposing NE in every subgame **eliminates moves that are not sequentially rational.**

SPNE: Backward induction

For the types of dynamic games we consider, there is a **practical way to solve for the SPNE of the game.**

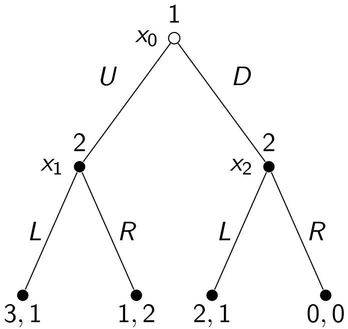
Instead of looking for a NE in every subgame, we can also **start from each terminal history** of the game and go backward while checking for optimal moves.

This is what we call **backward induction.**

- It is **not** another solution concept.
- It is a **practical way** to find SPNE.

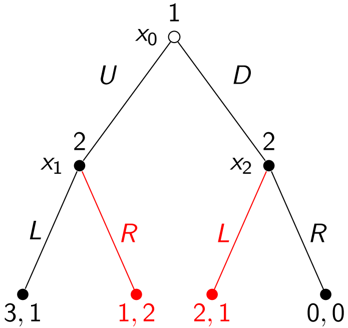
Backward induction: Example 1

Consider once again this example.



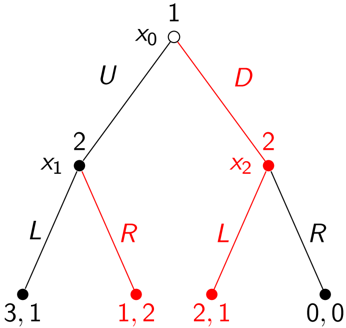
Backward induction: Example 1

Starting *from the end*, we find the best choice of P2 in the subgames starting at x_1 and x_2, respectively.



Backward induction: Example 1

Then, we go backward, to x_0 and we determine P1 best choice given the future choices of P2.



Backward induction: Example 1

Backward induction yields that (D, RL) is a SPNE.

The other Nash equilibrium, (U, RR) **did not survive the subgame-perfect refinement.**

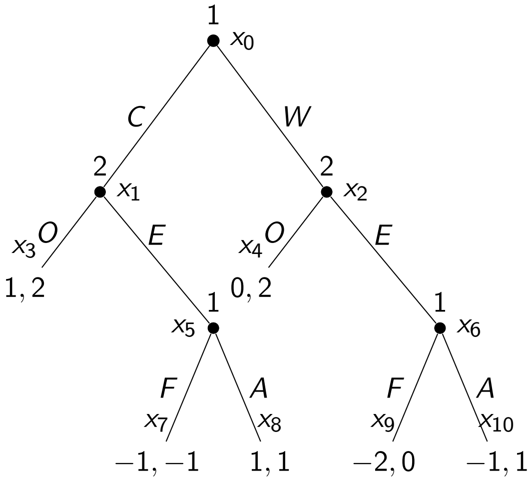
- It is the case because R is not a NE of the subgame starting at x_2.

Once again, it is important to **include all strategies in the description of the SPNE** (D, RL).

- Knowing that P2 plays R after U is as important as knowing that they play L after D.

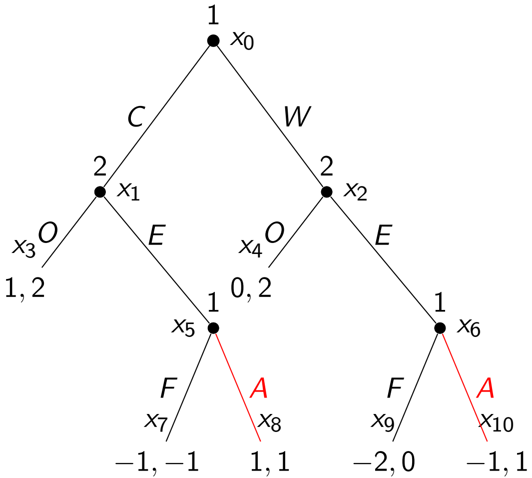
Backward induction: Example 2

Consider this more complicated example.



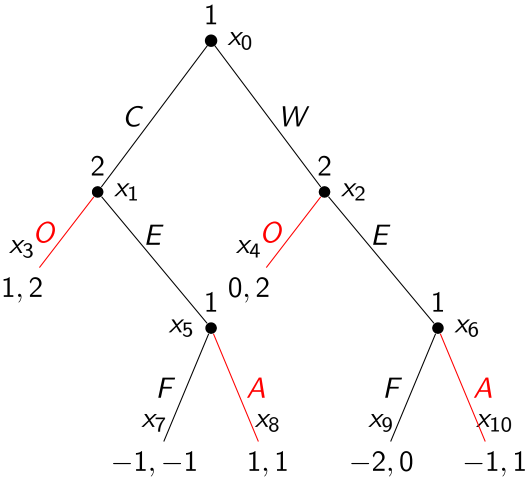
Backward induction: Example 2

Step 1 of using backward induction yields:



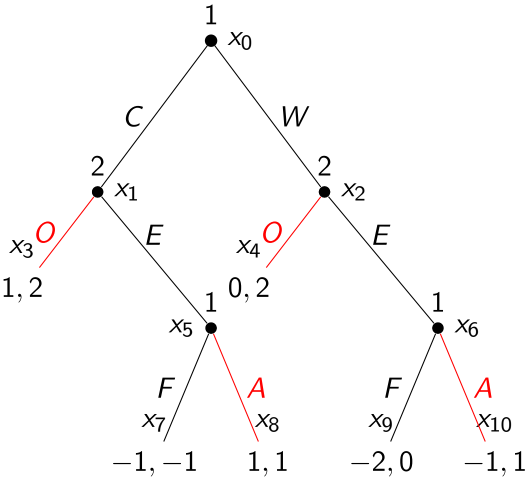
Backward induction: Example 2

Step 2 of using backward induction yields:



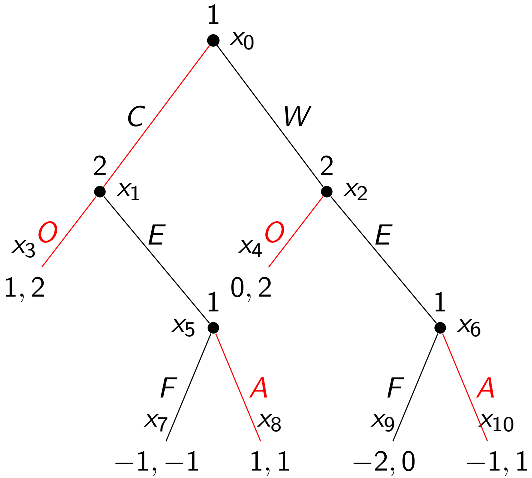
Backward induction: Example 2

Step 3 of using backward induction yields:



Backward induction: Example 2

Step 3 of using backward induction yields:



Backward induction: Example 2

Collecting all the strategies that we have identified in each subgame yields that:

The strategy profile (CAA, OO) is a SPNE of the game.

We can check that this strategy profile contains strategies that are NE in every subgames.

Backward induction: Example 2

Let us start with the game itself.

1 \ 2	OO	OE	EO	EE
CFF	$1, 2$	$1, 2$	$-1, -1$	$-1, -1$
CFA	$1, 2$	$1, 2$	$-1, -1$	$-1, -1$
CAF	$1, 2$	$1, 2$	$1, 1$	$1, 1$
CAA	$1, 2$	$1, 2$	$1, 1$	$1, 1$
WFF	$0, 2$	$-2, 0$	$0, 2$	$-2, 0$
WFA	$0, 2$	$-1, 1$	$0, 2$	$-1, 1$
WAF	$0, 2$	$-2, 0$	$0, 2$	$-2, 0$
WAA	$0, 2$	$-1, 1$	$0, 2$	$-1, 1$

Backward induction: Example 2

Underlining best responses:

$1 \backslash 2$	OO	OE	EO	EE
CFF	$\underline{1}, \underline{2}$	$\underline{1}, \underline{2}$	$-1, -1$	$-1, -1$
CFA	$\underline{1}, \underline{2}$	$\underline{1}, \underline{2}$	$-1, -1$	$-1, -1$
CAF	$\underline{1}, \underline{2}$	$\underline{1}, \underline{2}$	$\underline{1}, 1$	$\underline{1}, 1$
CAA	$\underline{1}, \underline{2}$	$\underline{1}, \underline{2}$	$\underline{1}, 1$	$\underline{1}, 1$
WFF	$0, \underline{2}$	$-2, 0$	$0, \underline{2}$	$-2, 0$
WFA	$0, \underline{2}$	$-1, 1$	$0, \underline{2}$	$-1, 1$
WAF	$0, \underline{2}$	$-2, 0$	$0, \underline{2}$	$-2, 0$
WAA	$0, \underline{2}$	$-1, 1$	$0, \underline{2}$	$-1, 1$

Backward induction: Example 2

We have **8 NE.**

All of them lead to the same *outcome* but we know that 7 of them have some **strategies that are not sequentially rational.**

Let us investigate now the subgame starting at x_1 in normal-form:

$1 \backslash 2$	O	E
F	$1, 2$	$-1, -1$
A	$1, 2$	$1, 1$

Backward induction: Example 2

Hence we obtain the following NE:

1 \2	O	E
F	$\underline{1}, \underline{2}$	$-1, -1$
A	$\underline{1}, \underline{2}$	$\underline{1}, 1$

And if we check at the tree, we see that playing F in the subgame starting at x_5 is not a NE of this subgame.

Hence, only (A, O) is a SPNE of the subgame starting at x_1.

Backward induction: Example 2

We can go the same for the subgame starting at x_2:

$1 \backslash 2$	O	E
F	$\underline{0}, \underline{2}$	$-2, 0$
A	$\underline{0}, \underline{2}$	$\underline{-1}, 1$

For the subgame starting at x_6, F is not a NE of this subgame.

Hence, only (A, O) is a SPNE of the subgame starting at x_2.

Backward induction: Example 2

In the two subgames starting at x_1 and x_2, only (A, O) is a SPNE.

Then the only NE **that survives subgame perfection** is (CAA, OO).

It is the **same strategy profile** as the one we have found using **backward induction.**

Table of Contents

Stackelberg duopoly

We now investigate the **Stackelberg duopoly** setting.

Consider two firms, 1 and 2, producing an **homogeneous good**.

Firms choose how much quantity of the good they produce.

- Let q_1 and q_2 denote those quantities.
- The market price is defined by $P(q_1 + q_2)$ with $P' < 0$ and P continuously differentiable.

Each firm has a **marginal cost of production** equal to c_i.

Stackelberg : Normal form

Let us write this game using the normal-form representation.

- **1. Players:** Firms, $N = \{1, 2\}$.
- **2. Actions:** Each firm chooses its quantity $q_i \in \mathbb{R}_+$.
  - Hence the action space of firm i is $A_i = \mathbb{R}_+$.
- **3. Payoffs:** Profits $\pi_i(q_i, q_j) = \big(P(q_i + q_j) - c_i\big)q_i$.

So far, **identical to the Cournot setting.**

Stackelberg duopoly: Dynamics

The **difference** between the **Cournot and the Stackelberg** settings is the following.

In Stackelberg, we assume that one firm, say 1, **plays first** and set q_1. The second firm **observes** q_1 and then decide how much to produce, q_2.

We say that firm 1 is the **leader** while firm 2 is the **follower.**

Hence, the Stackelberg duopoly is a **dynamic game.**

Stackelberg duopoly: Backward induction

We proceed by **backward induction** to solve this problem.

That is, we start in the **subgame in which firm 1 has already set its quantity** to q_1 and we find firm 2's best-response function to that q_1.

Once we have firm 2's best-response function, **we go back to the first period** of the game and we determine firm 1's optimal choice of q_1 given firm 2's best-response function.

Stackelberg duopoly: Backward induction

Assume that the price is given by $P(q_1 + q_2) = a - b(q_1 + q_2)$.

Let us start in the **subgame** after which firm 1 has chosen to produce q_1.

Firm 2 takes q_1 as given and chooses q_2 so as to solve

$$\max_{q_2} \ \pi_2(q_1, q_2) = (a - b(q_1 + q_2) - c_2)q_2.$$

Notice that this is the **exact same problem faced by firm 2** in the Cournot setting.

Stackelberg duopoly: Backward induction

Hence, we expect firm 2's **best-response function** to q_1 to be **exactly the same** as in the Cournot setting, that is,

$$q_2(q_1) = \frac{a - c_2 - bq_1}{2b}.$$

The subtle difference with the Cournot case is that this best-response function is actually the best-response **to what firm 2 observes**.

Stackelberg duopoly: Backward induction

Now we can **go back to the first stage**, when firm 1 plays.

Firm 2 perfectly anticipates that if it produces q_1, then firm 2 will produce $q_2(q_1)$.

Firm 1 has no doubt about it because by definition $q_2(q_1)$ is the best thing firm 2 can do when facing q_1.

Stackelberg duopoly: Backward induction

Hence, firm 1 does not consider q_2 as given, but consider firm 2's quantity to **explicitly depends on** q_1 through $q_2(q_1)$.

Firm 1's problem therefore writes as follows:

$$\max_{q_1} \ \pi_1(q_1, q_2) = \big(a - b\big(q_1 + q_2(q_1)\big) - c_1\big)q_1.$$

This maximization problem shows that firm 1 **completely internalizes** firm 2's best response.

Firm 1's problem **depends only on** q_1 (not on q_2), hence solving it will give us the equilibrium quantity q_1^* directly.

Stackelberg duopoly: SPNE

In practice, you replace firm 2's best-response function $q_2(q_1)$ by its expression and you obtain the following problem for firm 1:

$$\max_{q_1} \ \pi_1(q_1, q_2) = \big(a - b\big(q_1 + \frac{a - c_2 - bq_1}{2b}\big) - c_1\big)q_1.$$

Solving this problem (use the FOC) we directly obtain **firm 1's equilibrium quantity**:

$$q_1^* = \frac{a + c_2 - 2c_1}{2b}.$$

Stackelberg duopoly: SPNE

To obtain **firm 2's equilibrium quantity**, we only have to plug q_1^* into firm 2's best-response function:

$$\begin{aligned} q_2^* = q_2(q_1^*) &= \frac{a - c_j - bq_1^*}{2b} \\ &= \frac{a - c_2}{2b} - \frac{1}{2}\frac{a + c_2 - 2c_1}{2b} \\ &= \frac{a + 2c_1 - 3c_2}{4b}. \end{aligned}$$

We can say that (q_1^*, q_2^*) is the SPNE of the Stackelberg duopoly.

Stackelberg duopoly: Remarks

One can compute the total equilibrium quantity and equilibrium price in Stackelberg.

Total **quantity appears to be larger** in Stackelberg than in Cournot.

And so the **market price is lower** in Stackelberg than in Cournot.

Finally, the leader **obtains a higher profit** than the follower in Stackelberg.

- Sometimes called the **first-mover advantage.**

Table of Contents

Observability in dynamic games

The extensive form can also be used to represent static games.

What really defines a static game is not the absence of dynamic but the **inability for players to observe the action taken by the other players.**

Consider a game with two players in which player 1 "physically" plays before player 2.

- If player 2 **does not observe** the action taken by player 1, it is *as if* the two players were **playing simultaneously.**
- In other words, it is *as if* the game were a static game.

Dynamics without observability

Consider the following static game of complete information.

1 \2	L	R
U	1, 3	3, 2
D	4, 0	2, 2

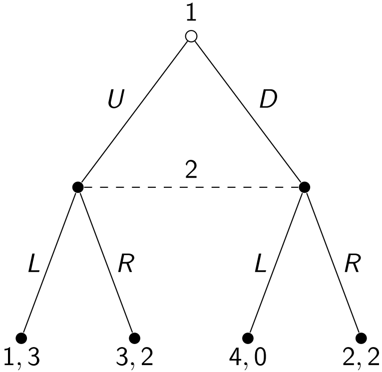
So far, we have interpreted this game as players simultaneously choosing an action.

But this game could also be interpreted as follows:

- 1. Player 1 first **secretly** chooses U or D.
- 2. Player 2 chooses L or R.

Extensive form of a static game

We could represent this static game with in extensive form.



Where the dashed line represents player 2's inability to distinguish at which node their are when playing.

Extensive form of a static game

Notice that we could also change the order in which players choose their action and still represent the same static game.