

GAME THEORY: DYNAMIC GAMES OF INCOMPLETE INFORMATION

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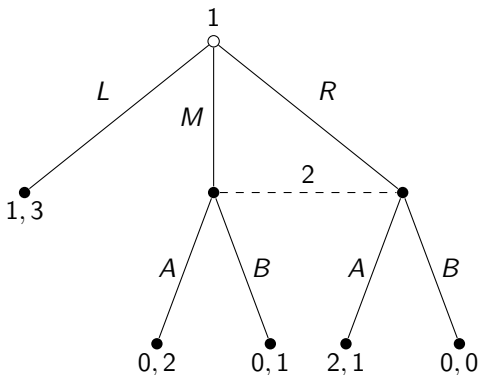
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Introducing example

Consider the following game inspired by Selten (1975):

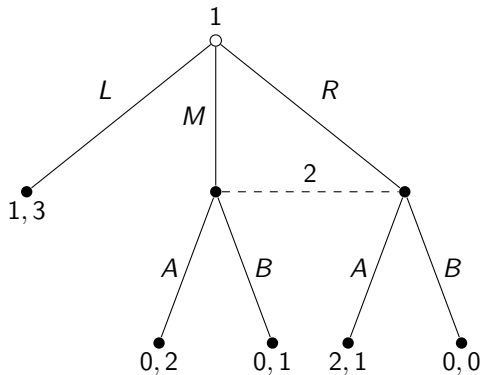


Player 2 observes whether player 1 played L or not.

▷ But cannot distinguish M from R

Introducing example

It can be seen as a dynamic game of **imperfect** information.

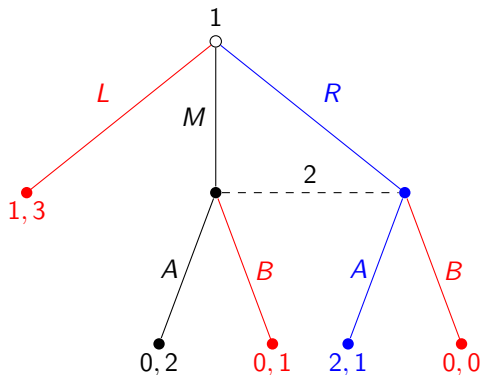


$1 \backslash 2$	A	B
L	1,3	1,3
M	0,2	0,1
R	2,1	0,0

Introducing example

There are **two pure-strategy Nash Equilibria** in this game:

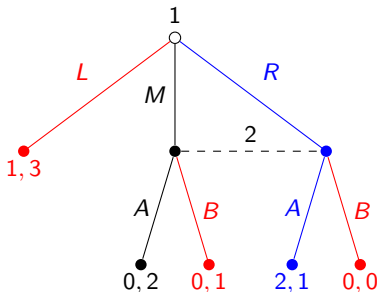
▷ (L, B) and (R, A)



$1 \backslash 2$	A	B
L	$1, \underline{3}$	<u>$1, 3$</u>
M	$0, \underline{2}$	$0, 1$
R	<u>$2, 1$</u>	$0, 0$

Introducing example

Do you find the pure-strat. NE (L, B) *satisfactory*?



In (L, B) , P2 *threatens* P1 to play B if the latter chooses M or R .

- ▷ B is a **non-credible threat**
- ▷ If P1 plays M or R instead, P2 would prefer A in both cases (dominant strat)

Introducing example

(L, B) is a well-defined pure-strategy NE but we do not really like it.

- ▷ Seems *implausible*
- ▷ It is **not sequentially rational**

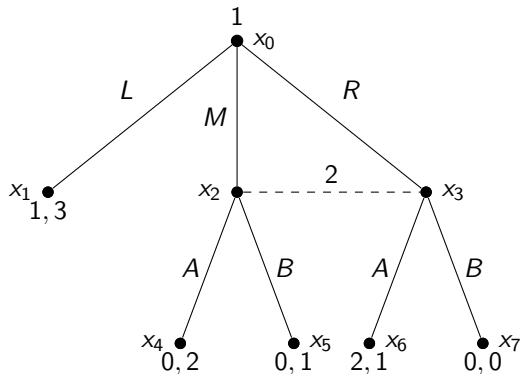
But it is a **dynamic game**.

- ▷ We have already seen that NE is not a *satisfactory* equilibrium concept for dynamic games.

That is why we introduced the notions of **subgames** and **subgame-perfect NE**.

- ▷ Let us apply this concept in the example!

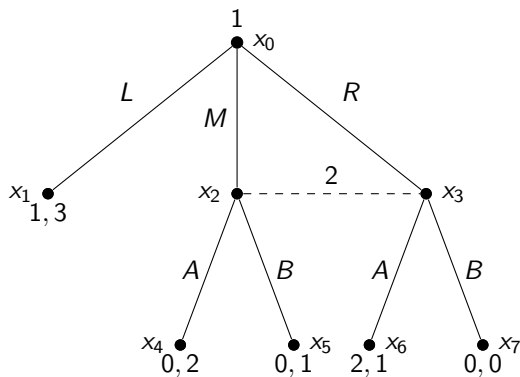
Introducing example



Extensive form:

- $N = \{1, 2\}$
- $A_1 = \{L, M, R\}$, $A_2 = \{A, B\}$
- $X_1 = \{x_0\}$, $X_2 = \{x_2, x_3\}$
- $I_1 = \{\{x_0\}\}$, $I_2 = \{\{x_2, x_3\}\}$
- $r = \{x_0\}$
- $T = \{x_1, x_4, x_5, x_6, x_7\}$

Introducing example



Extensive form:

- $N = \{1, 2\}$
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- $T = \{x_1, x_4, x_5, x_6, x_7\}$

\Rightarrow Only **one** subgame:
▷ The game itself!

Introducing example

Applying the **refinement** of subgame perfection **to this game**:

- ▷ Subgame-perfect NE \Leftrightarrow Nash Equilibrium

It means that the subgame perfection refinement **has no bite** in this example.

- ▷ Applying it does not help removing the *unsatisfactory* Nash equilibrium (L, B) .

We have to create another **refinement** to tackle this issue.

- ▷ It will be the **Perfect Bayesian Equilibrium**

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Preliminaries

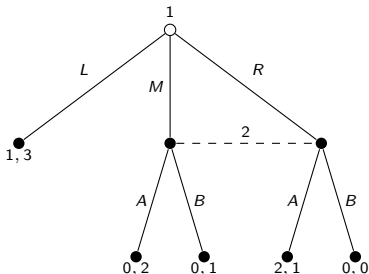
We would like to *get rid of* (L, B) .

The idea is to restore the notion of **non-credible threat** to *improper subgames*.

We must then find a way for P2 to **distinguish** one node from another even when their information set does not allow it.

Beliefs are the key.

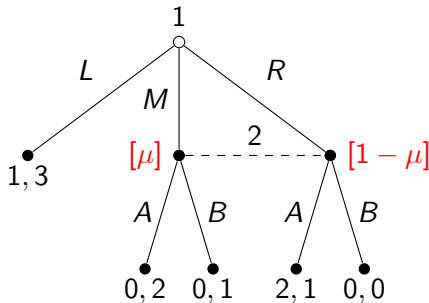
- ▷ We will allow P2 to **form beliefs on the probability** that M and R have been played by P1.
- ▷ And therefore to have a **strategy that depends on those beliefs**.



Preliminaries

For instance, assume that if P1 does not play L.

- ▷ P2 believes that M occurs with probability $\mu \in [0, 1]$ and R with probability $1 - \mu$.



Preliminaries

P2's **expected payoff** when playing:

▷ $A: \mu \cdot 2 + (1 - \mu) \cdot 1 = \mu + 1$

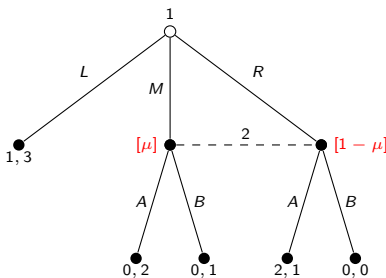
▷ $B: \mu \cdot 1 + (1 - \mu) \cdot 0 = \mu$

For any $\mu \in [0, 1]$:

▷ P2 prefers to play A

Whatever P2's the belief,
B is not a best-response anymore.

This is enough to get rid of (L, B)



Preliminaries

Allowing P2 to have **beliefs on indistinguishable nodes** of their information set “solves” our problem of **non-credible threats**.

Natural questions:

1. Is it **reasonable** to assume that players have beliefs on indistinguishable nodes?
2. Where are those beliefs coming from?

Both questions will be answered by the new equilibrium concept.

- ▷ Key-feature: Beliefs will now be **considered part of the equilibrium**.
- ▷ Beliefs will emerge **endogenously** together with strategies.

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Beliefs and sequential rationality

Previous example: Useful to identify the **failure of subgame perfection** when the information set of a player is **not a singleton**.

Our goal is to define a setting in which we can say things like:

- ▷ “Player i is not **sequentially rational**”
- ▷ At every node where i plays, even if the information set **is not a singleton**

And then **apply this refinement** to Bayesian Nash equilibria of the game to remove *unreasonable* ones.

- ▷ As we did with subgame perfection: Take all the NE of the game and keep only those surviving subgame perfection (i.e. that are sequentially rational).

Methodology

We want to **refine** the concept of BNE.

To this end, we are going to impose

- ▷ **four requirements** on beliefs.

Finally, we will **impose those requirements** on BNE strategy profiles.

- ▷ and we will obtain a new equilibrium concept: **Perfect BNE**.

Decision nodes and information sets

Notation:

- ▷ X_i denote the set of player i 's decision nodes
- ▷ H_i denote the set of player i 's information sets
 - ▷ It is a partition of X_i

Example: Assume that in some extensive-form game player i 's decision nodes are in $X_i = \{x_1, x_2, x_4, x_6\}$.

H_i can be any partition of X_i , for instance:

- ▷ $H_i = \{\{x_1\}, \{x_2\}, \{x_4\}, \{x_6\}\}$: All singletons
- ▷ $H_i = \{\{x_1, x_2\}, \{x_4\}, \{x_6\}\}$: x_1 and x_2 are not distinguishable
- ▷ $H_i = \{\{x_1, x_2, x_4\}, \{x_6\}\}$: only x_6 or “not x_6 ” is distinguishable
- ▷ $H_i = \{\{x_1, x_2, x_4, x_6\}\}$: nothing is distinguishable

System of beliefs

Definition: In an extensive-form game, a **system of beliefs** μ is a probability distribution over decision nodes within each information set.

Formally, for every player $i \in N$, every information set $h \in H_i$ and every of its decision node $x \in h$, $\mu(x) \in [0, 1]$ is the probability that player i assigns to decision node x when player i moves to information set h .

Where $\sum_{x \in h} \mu(x) = 1$ for every $h \in H_i$, $i \in N$.

Example: Take $H_i = \{\{x_1, x_2, x_4\}, \{x_6\}\}$, $h_1 := \{\{x_1, x_2, x_4\}\}$ and $h_2 := \{\{x_6\}\}$ then the system of beliefs **may** assign:

$$\triangleright \mu(x_1) = 2/3, \mu(x_2) = 1/6, \mu(x_4) = 1/6 \Rightarrow \sum_{x \in h_1} \mu(x) = 1$$

$$\triangleright \mu(x_6) = 1$$

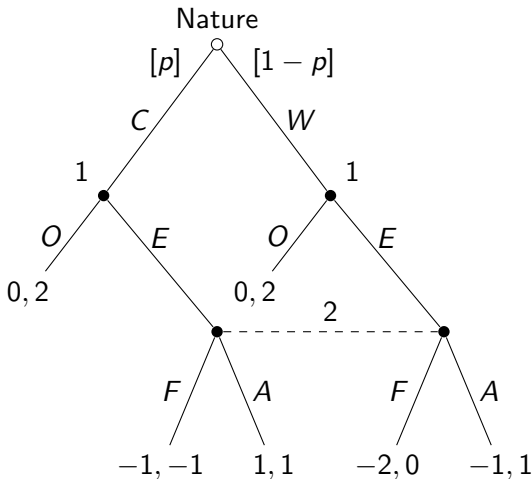
Beliefs: Requirement 1

Requirement 1: Every player has well-defined beliefs over their decision nodes at every of their information set (singleton or not)

That is, the game is endowed with a **complete system of beliefs**

Requirement 1: An Example

Consider the following game (Tadelis, 2013):



Requirement 1: An Example

A system of beliefs
must assign a probability
to x_1 , x_2 , x_3 and x_4 .

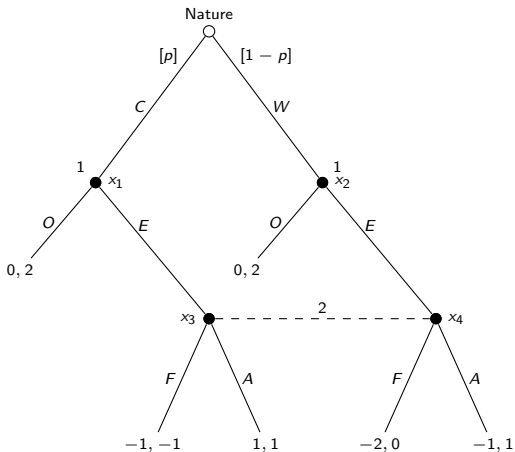
We have

$$\triangleright \mu(x_1) = \mu(x_2) = 1$$

$$\triangleright \mu(x_3) \in [0, 1]$$

$$\triangleright \mu(x_4) \in [0, 1]$$

$$\triangleright \mu(x_3) + \mu(x_4) = 1$$



Beliefs: Where do they come from?

We have imposed a **system of beliefs** but how are they determined?

- ▷ Are they imposed by exogenous elements?
- ▷ Can players “choose” their beliefs?

We are going to allow for both in some way.

- ▷ **Exogenously:** Beliefs are partly determined by Nature.
- ▷ **Endogenously:** Beliefs are partly determined by **players' strategies**.

Beliefs: Consistency constraints

We impose some **consistency constraints** on beliefs.

- ▷ **Exogenously:** Beliefs must be consistent with Bayes' rule (we will be more specific later).
- ▷ **Endogenously:** Beliefs must be consistent with how we anticipate other players' strategies.

Reminder: Bayes' rule

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space

For any two events $A, B \in \mathcal{F}$ such that $\mathbb{P}(B) \neq 0$ we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)}$$

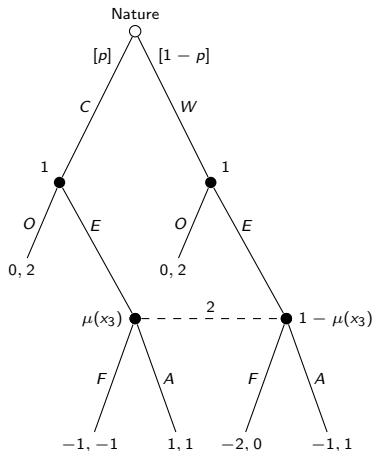
Beliefs: Consistency constraints

Go back to the example

- ▷ Constraints on $\mu(x_3)$?

Assume P1 plays EO , i.e.:

- ▷ When P1 is C : chooses E
- ▷ When P1 is W : chooses O



Beliefs: Consistency constraints

Go back to the example

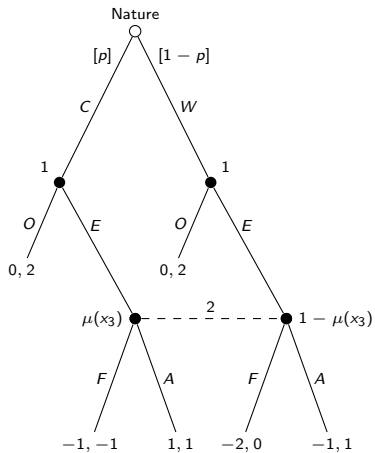
- ▷ Constraints on $\mu(x_3)$?

Assume P1 plays EO , i.e.

- ▷ When P1 is C : chooses E
- ▷ When P1 is W : chooses O

Belief consistency (endogenous):

- ▷ $\mu(x_3) = \mathbb{P}(\text{P1 is } C \mid E)$
- ▷ $1 - \mu(x_3) = \mathbb{P}(\text{P1 is } W \mid E)$

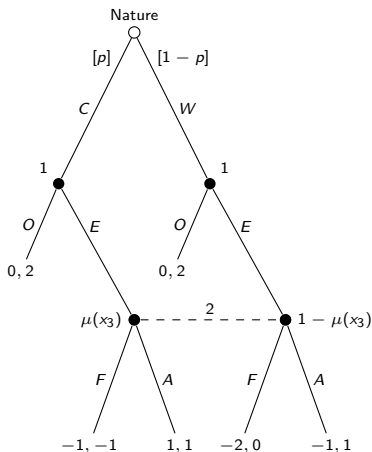


Beliefs: Consistency constraints

Therefore if P1 plays *EO* we must have:

- ▷ $\mu(x_3) = 1$.
- ▷ If P2 observes that the game reached this stage, it must be that P1 is **not** *W*.

P2's beliefs must be **consistent** with what P2 thinks P1 will play.



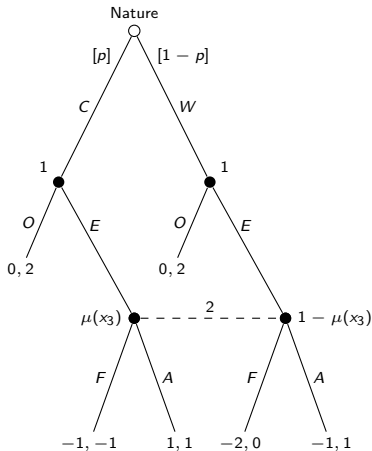
Beliefs: Consistency constraints

This also means that if $P1$ considers playing EO

- ▷ Anticipates that $\mu(x_3) = 1$.
- ▷ Can therefore anticipate that $P2$ **will play A following E** .

Then when $P1$ considers a **deviation** from EO .

- ▷ $P1$ could try to play E when W .
- ▷ $P2$ would **wrongly** believe that $P1$ is C and would play A .
- ▷ $P1$ therefore knows that this **deviation would not be profitable**



Beliefs: Consistency constraints

The above example illustrates an **endogenous consistency** requirement.

- ▷ When a player reaches a decision node for which the information set is not a singleton it must form a belief for each decision node that is **consistent** with the other players' strategies.

But it must also be **consistent with exogenous elements** such as Nature draw.

- ▷ We do not see it in the previous example
- ▷ See next slide

Beliefs: Consistency constraints

Consider the following case: P2 thinks that P1 of type

- ▷ C : chooses E with **probability** $\sigma_C \Leftrightarrow \mathbb{P}(E \mid C) = \sigma_C$
- ▷ W : chooses E with **probability** $\sigma_W \Leftrightarrow \mathbb{P}(E \mid W) = \sigma_W$

What should be $\mu(x_3)$?

- ▷ Assume P2 observes that P1 played E .
- ▷ Then, P2 must then infer how likely it is that P1 is C given that they played E .

Formally,

$$\mu(x_3) = \mathbb{P}(\text{Nature has chosen } C \mid \text{P1 played } E).$$

Beliefs: Consistency constraints

Using Bayes' rule we have that:

$$\begin{aligned}\mu(x_3) &= \mathbb{P}(\text{Nature has chosen } C \mid \text{P1 played } E) \\ &= \frac{\mathbb{P}(\text{Nature has chosen } C \text{ AND P1 played } E)}{\mathbb{P}(\text{P1 played } E)}.\end{aligned}$$

With lighter notations:

$$\mu(x_3) = \frac{\mathbb{P}(C \text{ AND } E)}{\mathbb{P}(E)}.$$

Beliefs: Consistency constraints

Using **Bayes' rule** once again and the **law of total probability** we have:

$$\begin{aligned}\mu(x_3) &= \frac{\mathbb{P}(C \text{ AND } E)}{\mathbb{P}(E)} \\ &= \frac{\mathbb{P}(E | C)\mathbb{P}(C)}{\mathbb{P}(E | C)\mathbb{P}(C) + \mathbb{P}(E | W)\mathbb{P}(W)} \\ &= \frac{\sigma_C \cdot p}{\sigma_C \cdot p + \sigma_W \cdot (1 - p)}\end{aligned}$$

Reminder:

- ▷ $\mathbb{P}(A, B) = \mathbb{P}(A | B)\mathbb{P}(B)$ for any two $A, B \in \mathcal{F}$
- ▷ $\mathbb{P}(A) = \sum_i \mathbb{P}(A | B_i)\mathbb{P}(B_i)$ where $(B_i)_{i=1}^m$ is a partition of \mathcal{F}

Beliefs: Consistency constraints

Notice that the belief

$$\mu(x_3) = \frac{\sigma_C \cdot p}{\sigma_C \cdot p + \sigma_W \cdot (1 - p)},$$

depends both on

- ▷ P1's strategy: **Endogenous** consistency
- ▷ Nature's draw: **Exogenous** consistency

Rational players form their beliefs using **both of these elements**.

Notice also that if we set $\sigma_C = 1$ and $\sigma_W = 0$.

- ▷ P1 chooses E when C and O when W with **certainty**.

- ▷ Then
$$\mu(x_3) = \frac{1 \cdot p}{1 \cdot p + 0 \cdot (1 - p)} = 1.$$

Equilibrium path: On and off

We are almost ready to state our **second and third requirements** on beliefs.

But first, consider the following definition.

Definition (Tadelis, 2013): Let $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ be a Bayesian Nash Equilibrium profile in a game of incomplete information. We say that an information set is **on the equilibrium path** if given σ^* and given the distribution of types, it is reached with **positive probability**.

By opposition, an information set is said to be **off the equilibrium path** if given σ^* , it is reached with **zero probability**.

Equilibrium path: Example

Example:

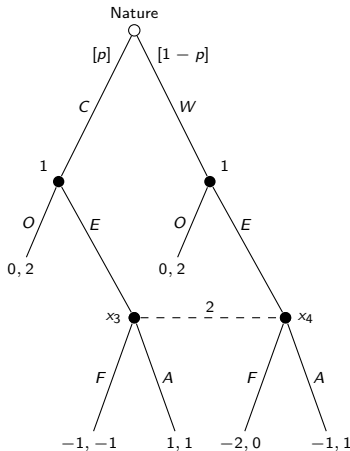
Consider first that P1 chooses EO .

- ▷ With prob. p , P1 is C and plays E .
- ▷ With prob. $1 - p$, P1 is W and plays O .

\Rightarrow The information set $h_1 = \{x_3, x_4\}$ is reached with positive probability p .

If EO

were part of a BNE, we would say that h_1 is **on the equilibrium path** of this BNE.



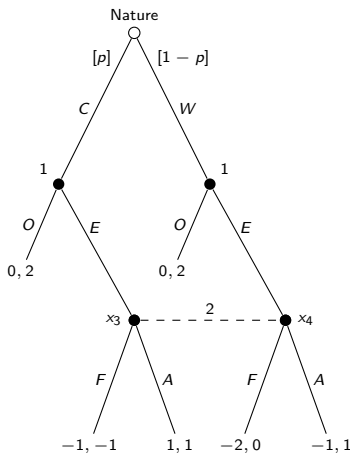
Equilibrium path: Example

Example: Consider now that P1 chooses OO

- ▷ With prob. p , P1 is C and plays O .
- ▷ With prob. $1 - p$, P1 is W and plays O .

⇒ The information set $h_1 = \{x_3, x_4\}$ is **never reached** with positive probability.

If OO were part of a BNE, we would say that h_1 is **off the equilibrium path** of this BNE.



Equilibrium path: On and off

Whether an information set is **on or off the equilibrium path** is not exogenous.

- ▷ It is determined by the players' actions.

We are now ready to state our **second and third requirements** on beliefs.

- ▷ One for beliefs **on** the equilibrium path.
- ▷ One for beliefs **off** the equilibrium path.

Beliefs: Requirement 2

Requirement 2 (Tadelis, 2013): For any BNE strategy profile σ^* , in all *information sets* that are **on** the equilibrium path, beliefs must be **consistent with Bayes' rule**.

That is, players must form their beliefs using both the

- ▷ **exogenous constraints** (nature);
- ▷ and the **endogenous constraints** (other players' strategies).

When Bayes is off path

What about information sets that are **off the equilibrium path**?

- ▷ Can't we **apply Bayes' rule** as well?
- ▷ Not always!

Recall that if P1's strategy is OO .

- ▷ Then $h_1 = \{x_3, x_4\}$ is **never reached**.

Assume that P2 **believes** that P1 plays OO .

- ▷ But surprisingly **observes** that $h_1 = \{x_3, x_4\}$ **is reached**!

Trying to apply Bayes' rule gives:
$$\mu(x_3) = \frac{0 \cdot p}{0 \cdot p + (1 - p) \cdot 0} = \frac{0}{0}!$$

When Bayes is off path

Clearly, applying Bayes' rule **fails** as $\mu(x_3) = \frac{0}{0}$ is **undefined**.

But you might wonder: **Why should we care?**

- ▷ It never happens at equilibrium, so why is that a problem?

When Bayes is off path

To see why, assume P2 believes that P1 plays OO so that Bayes' rule **does not apply to assign beliefs** to $h_1 = \{x_3, x_4\}$.

- ▷ P1 will compute their payoff with OO knowing that P2 will believe that h_1 is never reached.
- ▷ But if P1 wants to see if they could deviate from that and play EO for instance.
- ▷ Then, h_1 would be reached with positive probability.
- ▷ But as it is unexpected for P2, $\mu(x_3)$ is not defined by Bayes' rule.
- ▷ So that P1 is unable to know what will happen and to compute their payoff if they play E .

When Bayes is off path

Therefore, if $\mu(x_3)$ is undefined, we are unable to fully compute **how P1 could deviate** from OO .

That is why we will impose that there also exists beliefs over nodes of information sets that are **off the equilibrium path**.

We can now state our **third requirement**.

Beliefs: Requirement 3

Requirement 3 (Tadelis, 2013): At information sets that are **off the equilibrium path**, any belief can be assigned to which Bayes' rule does not apply.

In other words:

- ▷ If Bayes' rule can be applied: **Apply it!**
- ▷ Otherwise: $\mu(x)$ **can be anything in** $[0, 1]$ for x at an off equilibrium path information set.

Notice that there is room for some **arbitrary choices** here.

- ▷ We might obtain different solutions if we **choose different beliefs** off the equilibrium path!

Beliefs: Requirement 4

We can finally state our **fourth requirement** on beliefs.

Requirement 4 (Tadelis, 2013): Given their beliefs, players' strategies must be **sequentially rational**. That is, in every information set players will play a **best response** to their beliefs.

With this requirement, we restore the possibility to evaluate whether a move is **sequentially rational** or not at every information set, including those that contain more than one decision node.

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Perfect Bayesian Nash Equilibrium: Definition

Finally, we can define what is a **Perfect Bayesian Nash Equilibrium**.

Definition: A **Perfect Bayesian Nash Equilibrium** consists of a Bayesian Nash Equilibrium profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ together with a system of beliefs μ that satisfy Requirements 1,2,3 and 4.

In other words, a **PBNE** is a BNE such that players are **sequentially rational** at every information set.

PBNE: Beliefs and strategies

In BNE, beliefs were **purely exogenous**.

- ▷ Strategies **depended** on beliefs.
- ▷ But beliefs **were independent** of strategies.

The fundamental feature of PBNE is that beliefs and strategies are **both part of the equilibrium outcome**.

- ▷ Strategies **depend** on beliefs
- ▷ Beliefs **depend** $\left\{ \begin{array}{l} \text{on Nature (what is given)} \\ \text{on strategies (what other players might do)} \end{array} \right.$

Beliefs emerge **endogenously**.

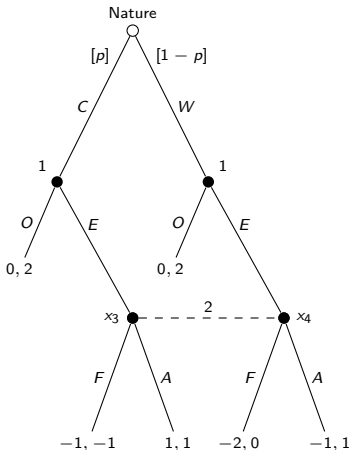
Perfect Bayesian Nash Equilibrium: An example

PBNE of this game with $p = 0.5$.

First, let us find the BNEs.

- ▷ We can compute the merged payoff matrix as follows.

1 \ 2	F	A
EE	$(-1, -2) ; -\frac{1}{2}$	$(1, -1) ; 1$
EO	$(-1, 0) ; \frac{1}{2}$	$(1, 0) ; \frac{3}{2}$
OE	$(0, -2) ; 1$	$(0, -1) ; \frac{3}{2}$
OO	$(0, 0) ; 2$	$(0, 0) ; 2$



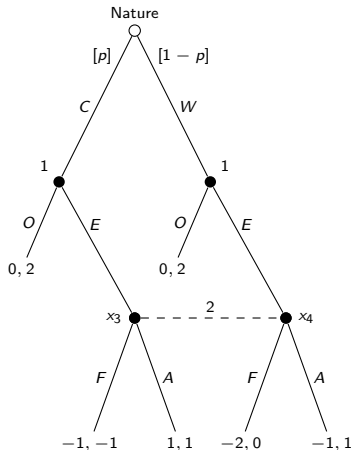
PBNE: An example

PBNE of this game with $p = 0.5$

First, let us find the BNEs

- ▷ We can compute the merged payoff matrix as follows

1 \ 2	F	A
EE	$(-1, -2) ; -\frac{1}{2}$	$(\underline{1}, -1) ; \underline{1}$
EO	$(-1, \underline{0}) ; \frac{1}{2}$	$(\underline{1}, \underline{0}) ; \frac{3}{2}$
OE	$(\underline{0}, -2) ; \underline{1}$	$(0, -1) ; \frac{3}{2}$
OO	$(\underline{0}, \underline{0}) ; \underline{2}$	$(0, \underline{0}) ; \underline{2}$



PBNE: An example

Two BNE: (OO, F) and (EO, A) .

Consider (OO, F) :

- ▷ The information set $h_1 = \{x_3, x_4\}$ is **off the equilibrium path**.
- ▷ It means that $\mu(x_3)$ can be anything in $[0, 1]$ (Requirement 3).

However, assume that for some reason P2 observes $E \Rightarrow h_1$ is reached.

For any $\mu(x_3) \in [0, 1]$, P2's expected payoff is:

- ▷ $\mu(x_3) \cdot (-1) + (1 - \mu(x_3)) \cdot 0 = -\mu(x_3)$ if P2 plays F
- ▷ $\mu(x_3) \cdot 1 + (1 - \mu(x_3)) \cdot 1 = 1$ if P2 plays A

PBNE: An example

- ▷ $\mu(x_3) \cdot (-1) + (1 - \mu(x_3)) \cdot 0 = -\mu(x_3)$ if P2 plays F
- ▷ $\mu(x_3) \cdot 1 + (1 - \mu(x_3)) \cdot 1 = 1$ if P2 plays A

Then it is clear that P2 **will play** A for any value of $\mu(x_3)$.

In other words, (OO, F) is such that

- ▷ P2 is **not sequentially rational** (Requirement 4).

The BNE profile (OO, F) **does not survive** the PBNE refinement.

PBNE: An example

Consider now (EO, A) :

- ▷ The information set $h_1 = \{x_3, x_4\}$ is **on the equilibrium path**.
- ▷ Belief $\mu(x_3) = 1$ as only C chooses E .

P2 is then **certain** that observing E means that P1 is C .

So if P2 reaches h_1 .

- ▷ **Best response is A .**

Are we done?

PBNE: An example

Are we done?

- ▷ Not yet, we also have to verify that ***EO* is a best response to *A* and belief $\mu(x_3) = 1$.**

Fix *A* and $\mu(x_3) = 1$.

1. P1 deviates to *EE*

- ▷ P2 would always believe that P1 is *C*.
- ▷ P2 would then always play *A*.

Not sequentially rational for P1.

- ▷ When reaching x_2 , P1 knows that P2 will play *A*.
- ▷ Better playing *O*

PBNE: An example

Still fix A and $\mu(x_3) = 1$.

2. P1 deviates to OE

If P1 reaches x_1 and plays O .

- ▷ Not sequentially rational
- ▷ Would be better to play E so that P2 plays A

If P1 reaches x_2 and plays E .

- ▷ P2 will believe that P1 is C
- ▷ P2 will play A
- ▷ Not sequentially rational for P1 to play E at x_2

PBNE: An example

Still fix A and $\mu(x_3) = 1$.

3. P1 deviates to OO

If P1 reaches x_1 and plays O .

- ▷ Not sequentially rational
- ▷ Would be better to play E so that P2 plays A

PBNE: An example

Therefore:

- (EO, A) and $\mu(x_3) = 1$ is the **only PBNE** of the game.
- The other BNE profile is not sequentially rational according to our requirements.

PBNE: Introducing Example solution

Let us go back to the introducing example.

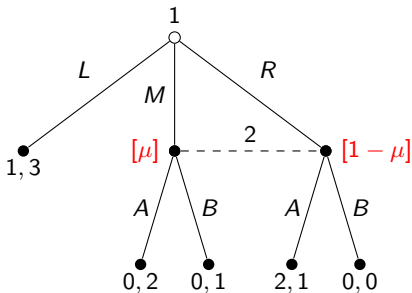
Recall: Two BNE.

▷ (L, B)

▷ (R, A)

We have found that for any $\mu \in [0, 1]$.

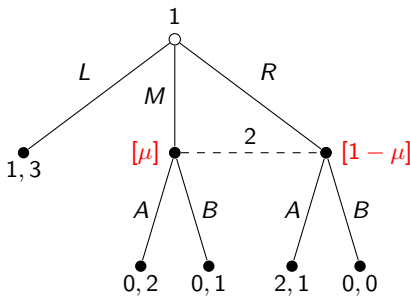
▷ A is a dominant strategy if we reach P2's information set.



PBNE: Introducing Example solution

Consider (L, B) .

- ▷ Information set is off the equilibrium path.
- ▷ Belief μ can be anything (requirement 3).
- ▷ But for any μ , A is a dominant strategy.



Then B is not sequentially rational.

- ▷ (L, B) is not a PBNE.

PBNE: Introducing Example solution

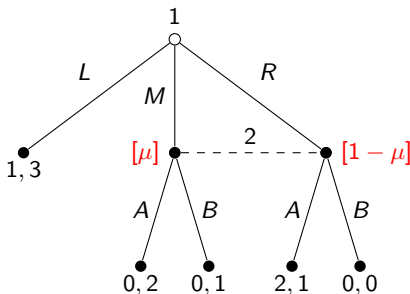
Consider (R, A) .

- ▷ Information set is on the equilibrium path.
- ▷ A is a dominant strat.
- ▷ P1 prefers R to M .
- ▷ Belief must be $\mu = 0$.

P1 can play:

- ▷ L and obtain 1.
- ▷ R and obtain 2 (better).

Therefore, (R, A) and $\mu = 0$ is a PBNE.



Other refinements

There exists **other refinements** of BNE.

For instance, “**sequential equilibrium**” is the most famous other one.

- ▷ Due to Kreps and Wilson (1982)

Sequential equilibrium is stronger than PBNE.

- ▷ Essentially a PBNE with **more requirements** on beliefs that are off the equilibrium path .
- ▷ Every sequential equilibrium is a PBNE, but the reverse does not hold.

In many applications, the two are equivalent.

- ▷ We will restrict to those cases

Table of Contents

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4. Perfect Bayesian Nash Equilibrium
5. Signaling games

Signaling games

A very important type of dynamic games of incomplete information:

- ▷ **Signaling games.**

They have some **distinguishable features**, which are, informally:

- ▷ P1 privately knows **payoff-relevant** information for P2
- ▷ No way to **certify** the information
- ▷ P1 will try to **signal** their information through their action
- ▷ Signaling is possible when actions are **credible signals**

Examples: Education, advertising, war games and sometimes even biological evolution!

Signaling games: Main Setting

The main setting for (two-player) signaling games is as follows:

1. Nature chooses a type. Only P1 learns it. But both P1 and P2's payoff depends on it;
2. P1 has at least as many actions as they have types (rich action space). Each action has some **cost**;
3. **Timing**: P1 plays first. P2 observes P1's action (but not type) and responds to it;
4. P2 updates their beliefs about P1's type thanks to their belief about P1's strategy and the observed action.

Signaling games: Classes of PBE

There are two important classes of PBE in signaling games.

1. Pooling equilibria

- ▷ All types of P1 choose the same action, i.e., P1 *pools together* all their types in the same action.
- ▷ P2 **cannot infer** anything about P1's type as their action is **non-revealing**.
- ▷ P2 must then best respond using only **exogenous information** about P1's type.

Signaling games: Classes of PBE

The other class is

2. Separating equilibria

- ▷ Each of P1's type chooses a **different action**.
- ▷ P2 can **perfectly infer** P1's type from their action.
- ▷ P2 can then respond *as if* they were perfectly informed about P1's type.

In that case, we say that P1's action **reveals** their type

Signaling games: Pooling and Separating

Separating equilibria seem very powerful.

- ▷ P1 cannot provide **hard proof** of their type but can only send a **signal**.
- ▷ Yet, P2 becomes **perfectly informed**.
- ▷ All information is **revealed**!

Signaling games: Pooling and separating

Separating seems **too good to be true**. What could go wrong?

- ▷ Recall that both P1 and P2's payoff depend on P1's type
- ▷ Maybe they do not have **aligned interests**?
- ▷ When P2 learns the information, they may take an action that does not please P1
- ▷ P1 might then have an **incentive to lie/manipulate information** so that P2 responds in a way that P1 prefers
- ▷ But a **rational P2 will anticipate** this possibility
- ▷ So if P1 has an incentive to lie, **P2 should not believe that P1's action reveals their type**

Signaling games: Pooling and Separating

The **existence** of a separating equilibrium therefore **relies** on the **credibility of P1's signal**.

If both players' interests are aligned.

- ▷ P1's signal is **always credible**.

If both players' interests are **not aligned**.

- ▷ We must check that P1 **does not want to manipulate information**.
- ▷ This will crucially rely on whether **sending wrong signals is costly** enough for P1.

Separating and Pooling: Previous example

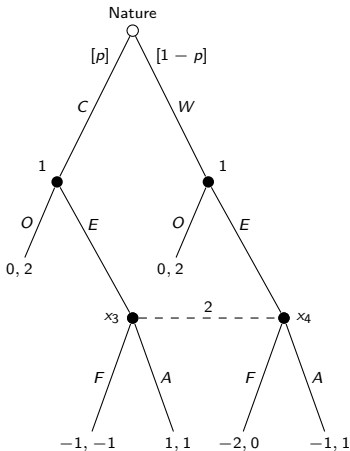
In the previous example:

The BNE: (OO, F) is a **pooling equilibrium**.

- ▷ Choice of $P1$ gives no information of their type.

The PBNE: (EO, A) is a **separating equilibrium**.

- ▷ If $P1$ plays E : $P2$ can perfectly infer that $P1$ is C .
- ▷ If $P1$ plays O : $P2$ can perfectly infer that $P1$ is W .



A famous game: Education as a signal

Famous signaling game: Education game **proposed by Spence (1973)**.

Spence's idea is that education is a **signal of productivity**.

- ▷ Job recruiters **cannot observe** workers' productivity.
- ▷ An individual can spend time and effort studying to get a diploma.
- ▷ It is **less costly** to get the diploma for **more productive** individuals.

Additional assumption: Education has no effect on productivity.

- ▷ i.e., education is nonproductive, only a loss of time and efforts.
- ▷ seems unrealistic but not a problem, assuming education is productive would not change the result.

A famous game: Education as a signal

Big picture:

- ▷ Investing in education is **very** costly for individuals with a low level of productivity.
- ▷ We expect that **only** highly productive individuals invest in education.
- ▷ Therefore, **education signals productivity**.

The obvious problem to this reasoning is

- ▷ Low types must not be incentivized to get the diploma to pretend they are high types.

A famous game: Education as a signal

Setting (Tadelis, 2013, p.319):

Nature draws P1's type: With probability p , P1 is of type $t_1 = H$ (High); otherwise P1 is $t_1 = L$ (Low) with probability $1 - p$.

P1 plays first (after Nature):

- ▷ P1 is the future employee
- ▷ P1 can choose to study for an undergraduate degree U only or to continue studying to obtain a graduate degree D
- ▷ To obtain U : Individual cost is normalized to 0
- ▷ To obtain D : It costs $c_H = 2$ and $c_L = 5$ to type H and L , respectively

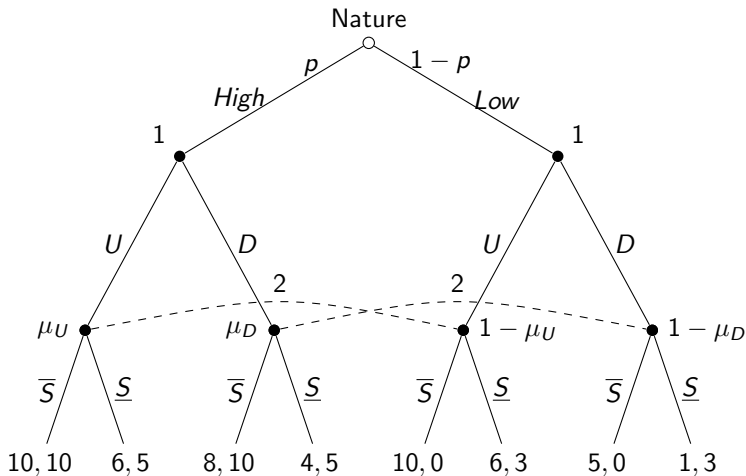
A famous game: Education as a signal

P2 observes $a_1 \in A_1 = \{U, D\}$ but not t_1

- ▷ P2 is the employer, plays after observing $a_1 \in A_1 = \{U, D\}$
- ▷ P2 must assign the employee to one of two possible tasks:
 $a_2 \in \{\underline{S}, \bar{S}\}$
- ▷ \underline{S} is *less skilled* task than \bar{S} : The market wage for performing \underline{S} is $\underline{w} = 6$ and the one for \bar{S} is $\bar{w} = 10$
- ▷ P2's net profit depends on the following task-productivity assignment (independent of U and D):

	\bar{S}	\underline{S}
H	10	5
L	0	3

A famous game: Education as a signal



A famous game: Education as a signal

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	\bar{S}	\underline{S}
H	10	5
L	0	3

A famous game: Education as a signal

We define a **system of beliefs**:

- ▷ μ_U : P2's belief that P1 is H after observing U .
- ▷ μ_D : P2's belief that P1 is H after observing D .

They will be determined both by

- ▷ Nature
- ▷ P1's strategy

A famous game: Education as a signal

First, let us find the BNEs when $p = \frac{1}{4}$.

There are **two pure-strategy BNE**:

- ▷ (UU, \underline{SS}) (Pooling)
- ▷ (DU, \overline{SS}) (Separating)

The proof is left as an exercise

- ▷ See Tadelis (2013), p.322

A famous game: Education as a signal

Consider the separating equilibrium $(DU, \bar{S}\underline{S})$

- ▷ All information sets are on the equilibrium path

$$\begin{aligned}\mu_U = \mathbb{P}(H \mid U) &= \frac{\mathbb{P}(U \mid H)\mathbb{P}(H)}{\mathbb{P}(U \mid H)\mathbb{P}(H) + \mathbb{P}(U \mid L)\mathbb{P}(L)} \\ &= \frac{0 \cdot p}{0 \cdot p + 1 \cdot (1 - p)} \\ &= 0\end{aligned}$$

When P2 observes U , they believe that P1 is L with certainty

A famous game: Education as a signal

Now for μ_D

$$\begin{aligned}\mu_D = \mathbb{P}(H \mid D) &= \frac{\mathbb{P}(D \mid H)\mathbb{P}(H)}{\mathbb{P}(D \mid H)\mathbb{P}(H) + \mathbb{P}(D \mid L)\mathbb{P}(L)} \\ &= \frac{1 \cdot p}{1 \cdot p + 0 \cdot (1 - p)} \\ &= 1\end{aligned}$$

When P2 observes D , they believe that P1 is H with certainty

A famous game: Education as a signal

For beliefs $\mu_U = 0$ and $\mu_D = 1$, it is clear that

- ▷ \bar{S} is a BR to D ($10 > 5$)
- ▷ \underline{S} is a BR to U ($3 > 0$)

For beliefs $\mu_U = 0$, $\mu_D = 1$ and $\bar{S}\underline{S}$:

- ▷ P1 of type H : Prefers D to U ($8 > 6$)
- ▷ P1 of type L : Prefers U to D ($6 > 5$)

Therefore $(DU, \bar{S}\underline{S})$ is a (separating) PBNE

A famous game: Education as a signal

Consider the pooling equilibrium (UU, \underline{SS})

- ▷ The information set for nodes after D is off the equilibrium path

For the one on the equilibrium path

- ▷ $\mu_U = p = \frac{1}{4}$
- ▷ that is, this belief for P2 is only constituted by the **exogenous information**
- ▷ Because P1 **reveals nothing** by playing U for each of their type

A famous game: Education as a signal

Consider the pooling equilibrium (UU, \underline{SS})

- ▷ Information set for nodes after D : off the equilibrium path
- ▷ Information set for nodes after U : on the equilibrium path

For the one **on** the equilibrium path

- ▷ $\mu_U = p = \frac{1}{4}$
- ▷ that is, this belief for P2 is only constituted by the **exogenous information**
- ▷ Because P1 **reveals nothing** by playing U for each of their type

A famous game: Education as a signal

For the one **off** the equilibrium path:

- ▷ μ_D can be anything between $[0, 1]$.

Let us compute P2's best response to observing D for belief $\mu_D \in [0, 1]$

- ▷ Playing \bar{S} : $10\mu_D + 0(1 - \mu_D) = 10\mu_D$
- ▷ Playing \underline{S} : $5\mu_D + 3(1 - \mu_D) = 2\mu_D + 3$

Therefore P1 prefers $\begin{cases} \bar{S} & \text{if } \mu_D \geq \frac{3}{8} \\ \underline{S} & \text{if } \mu_D \leq \frac{3}{8} \end{cases}$

A famous game: Education as a signal

Therefore, for \underline{S} to be a BR for P2 it must be the case that

$$\mu_D \leq \frac{3}{8}$$

This means that (UU, \underline{SS}) is a PBNE only if beliefs off the equilibrium path $\mu_D \in [0, \frac{3}{8}]$