Practice session 1

Game Theory - MSc EEBL

Guillaume Pommey
guillaume.pommey@uniroma2.eu

Uliana Sulakshina
sulakshina@gmail.com

September 19, 2024

Exercise 1. *Starters*

Consider the static game of complete information described by the following payoff matrix:

		Player 2		
		L	C	R
	T	(2,0)	(1,1)	(4,2)
Player 1	M	(3,4)	(1,2)	(2,3)
	B	(1,3)	(0,2)	(3,0)

1. Write the game as a normal-form game. That is, define the set of players N, the set of strategies S_i for each player $i \in N$, and the payoffs associated to each outcome of the game for each player $u_i(s)$ where $s = (s_1, \ldots, s_n) \in \bigtimes_{i \in N} S_i$.
2. Apply the concept of *Iterated Elimination of Strictly Dominated Strategies (IESDS)*, and write the strategies that survive this process (*Hint: Start with Player 1, then move to Player 2. Iterate*).
3. Find the *pure-strategy Nash equilibria (NE)* of this game.
4. Compare your findings in questions 2 and 3. Explain the relationship between *IESDS* and *NE*.

Exercise 2. ***Coordination, conflict and efficiency***

1. Consider the game of Exercise 1.

   (a) Find the Pareto efficient outcome(s).

   (b) Compare with the Nash equilibria obtained in question 3 (Exercise 1), what can you say?

2. Consider now the following game. Two friends, A and B, want to have a drink and there are $n \in \mathbb{N}$ bars open denoted by B_1, B_2, …, B_n. Unfortunately, A's cellphone is out of battery and they have not decided in which bar they wanted to go. The payoff matrix is as follows.

		Player 2				
		B_1	B_2	B_3	$\ldots$	B_n
	B_1	$(1,1)$	$(0,0)$	$(0,0)$	$\ldots$	$(0,0)$
	B_2	$(0,0)$	$(1,1)$	$(0,0)$	$\ldots$	$(0,0)$
Player 1	B_3	$(0,0)$	$(0,0)$	$(1,1)$	$\ldots$	$(0,0)$
	$\vdots$	$(0,0)$	$(0,0)$	$(0,0)$	$\ldots$	$(0,0)$
	B_n	$(0,0)$	$(0,0)$	$(0,0)$	$\ldots$	$(1,1)$

   (a) What can you say about the nature of the game? Is there any problem of conflicting interests? Of coordination?

   (b) Find the pure-strategy Nash equilibria.

   (c) Find the Pareto efficient outcome(s). Compare with the NE.

   (d) Assume A finds a plug to load their battery and can communicate with B, what could the two friends do?

3. Consider now the following game. Two coworkers, C_1 and C_2 are annoyed by a flickering light in their office. Each of them can report it by sending an email to the responsible person at a small, but positive individual cost $c > 0$. The payoff matrix is as follows.

		C_2	
		Report	Say nothing
C_1	Report	$(1-c, 1-c)$	$(1-c, 1)$
	Say nothing	$(1, 1-c)$	$(0,0)$

   (a) Characterize the pure-strategy Nash equilibria for all possible values of $c \in \mathbb{R}_+$.

(b) Assume $c \in (0,1)$. Find the Pareto efficient outcome(s). Compare with the NE when $c \in (0,1)$.

(c) What can you say about conflicting interests? And coordination? What is the importance of the individual cost, c, here? Comment.

4. Consider now the following game. Two pharmaceutical laboratories, P_1 and P_2, want to develop a new drug. They can either team up and aim at developing a new generation drug or each can work on its own to develop a less ambitious drug. However, they cannot work both on the common project and on the solo project so that if P_i works on the common project while P_j ($j \neq i$) works on the solo project, the common project will fail. The payoff matrix is as follows.

		P_2 Common	P_2 Solo
P_1	Common	$(2,2)$	$(0,1)$
	Solo	$(1,0)$	$(1,1)$

(a) Find the pure-strategy Nash equilibria.

(b) Find the Pareto efficient outcome(s). Compare with the NE.

(c) What can you say about conflicting interests? And coordination?

5. Consider now the following game. Consider two suspected individuals, A and B, which are interrogated, one by one, by a police officer. Each can decide to either stay quiet or to provide evidence to the police officer. If both individuals stay quiet, they are not convicted. If one provides evidence while the other stays quiet, the talkative one gets a sentence reduction. If both give evidence, they are both condemned to the highest possible sentence. The payoff matrix is as follows.

		B Quiet	B Talk
A	Quiet	$(1,1)$	$(-1,2)$
	Talk	$(2,-1)$	$(0,0)$

(a) Find the pure-strategy Nash equilibria.

(b) Find the Pareto efficient outcome(s). Compare with the NE.

(c) If the two individuals were able to communicate prior to the questioning by the police officer would they be able to reach the situation $\{\text{Quiet}, \text{Quiet}\}$? What if one could commit not to talk?

Exercise 3. *Study together*

Ann and Paul have to study for their Game Theory exam. They can decide to study at their own home or at the university library. If they both remain at home, Ann's payoff is 2 and Paul's payoff is 0; if they both study at the university, they study together and Ann's payoff is equal to Paul's payoff it is denoted by $x \in \mathbb{R}_+$; if only Ann goes to the library, Ann's payoff is 1 and Paul's payoff is -1; otherwise, if only Paul goes to the library, Ann's payoff is 2 and Paul's payoff is 1.

		Paul	
		Home	Library
Ann	Home	(,)	(,)
	Library	(,)	(x,x)

1. Fill the payoff matrix using the available information.

2. Write the game as a normal-form game.

3. Can "study at the library" be a dominant strategy for Ann? Can "study at home" be a dominant strategy for Ann? And for Paul? Explain (*Hint: this depends on the values taken by x*).

4. Characterize the Nash equilibria depending on the value of $x \in \mathbb{R}_+$. Is is possible to have "both go to the library" as a Nash Equilibrium?

Reminder.

- **Normal-Form Representation of Games.**

  A game in normal (or strategic) form has three elements:

  1. A set of players $N = \{1, \dots, n\}$ which we consider to be a finite set.
  2. Pure-strategy space S_i for each player $i \in N$. We note $S = \times_i S_i$ the strategy space and s_i an element of the set S_i.
  3. Payoff functions u_i for each player $i \in N$ where $u_i : S \to R$.

- **Dominant and Dominated Strategies.**

  Definition: The strategy $s_i \in S_i$ **strictly** dominates the strategy s_i' if:

  $$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}), \text{ for all } s_{-i} \in S_{-i}.$$

  Definition: The strategy s_i is **strictly** dominant if it **strictly** dominates s_i' for all $s_i' \neq s_i$.

  Definition: The strategy $s_i \in S_i$ **weakly** dominates the strategy s_i' if:

  $$\forall\, s_{-i} \in S_{-i} \quad u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}) \quad \text{(with at least one } s_{-i} \text{ that gives a strict inequality)}$$

- **Nash Equilibrium.**

  The strategy profile $s^* = (s_1^*, \dots, s_n^*)$ is a **Nash Equilibrium** if for all $i = 1, \dots, n$ and all $s_i \in S_i$ we have:

  $$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*).$$

- **Best response.**

  Definition: $s_i \in S_i$ is a **best response** to $s_{-i} \in S_{-i}$ if:

  $$u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i}), \text{ for all } s_i' \in S_i.$$

  Definition (more technical): the **best response correspondence** of player i is a correspondence $BR_i : S_{-i} \rightrightarrows S_i$ given by:

  $$BR_i(S_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

  Note: we use the term correspondence and not function because $BR_i(\cdot)$ is not necessarily single-valued. A function $f : X \to Y$ maps every element $x \in X$ to *one and only one point* $y \in Y$. A correspondence $g : X \rightrightarrows Y$, however, maps every element $x \in X$ to the *power set* of Y, namely 2^Y (*i.e.* the set of all subsets of Y).

Additional material. *Simple problems*

In this additional exercise, you can simply train yourself to find all the pure-strategy Nash equilibria in different games.

1. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(2,0)	(1,1)
	D	(3,4)	(1,2)

2. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(-1,3)	(-2,-1)
	D	(2,1)	(4,-3)

3. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(-2,5)	(-3,-1)
	D	(2,0)	(-2,3)

4. Find all pure-strategy Nash equilibria of the following game.

		Player 2	
		L	R
Player 1	U	(4,1)	(-7,-3)
	D	(8,-3)	(-2,5)

5. Find all pure-strategy Nash equilibria of the following game.

			Player 2	
		L	C	R
Player 1	U	(2,0)	(1,3)	(2,4)
	M	(3,4)	(1,2)	(0,3)
	D	(2,1)	(5,2)	(2,3)

6. Find all pure-strategy Nash equilibria of the following game.

		Player 2			
		A	B	C	D
Player 1	U	(3,3)	(1,0)	(0,3)	(2,2)
	M	(3,3)	(0,0)	(3,2)	(0,2)
	D	(4,1)	(2,2)	(2,0)	(3,1)