TU„T1„,

Automated Decision Making in Business and Economics

Financial Modeling and Valuation

Financial streams

Cash flow stream

An array of cash flows $x_1, \ldots, x_k, \ldots$, associated with an array of consecutive times $t_1, \ldots, t_k, \ldots$.

- Times t_k and cash flows x_k can be certain or random;
- Units: cash flows need a numeraire ($, €, gold oz.); times need time units (days, years)

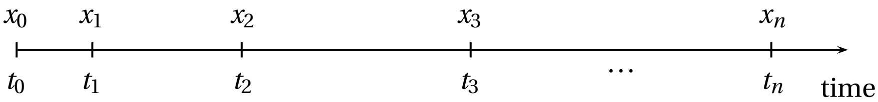

Figure: Graphical representation of a cash flow stream

Financial streams: discounting **one** period

Let r be the interest rate during the period $[0, 1]$.

- If you deposit \$1 at $t = 0$ you get $\$1 \cdot (1 + r)$ at $t = 1$.
- If you deposit S_0 at $t = 0$ you get $S_1 \doteq S_0 \cdot (1 + r)$ at $t = 1$.

Present Value

The **present value** at time $t = 0$ (today) of S_1 dollars at time $t = 1$ (tomorrow) is difined as

$$\frac{S_1}{(1 + r)},$$

i.e., the amount of money that should be deposited today to have exactly S_1 dollars tomorrow.

Financial streams: discounting **many** periods

Let r be the interest rate during the period $[0,1]$ and $n = 1, 2, \ldots$.

- If you deposit \$1 at $t = 0$ you get $\$1 \cdot (1+r)^n$ at $t = n$.
- If you deposit S_0 at $t = 0$ you get $S_n \doteq S_0 \cdot (1+r)^n$ at $t = n$.

Present Value

The **present value** at time $t = 0$ (today) of S_n dollars at time $t = n$ (after n periods) is defined as

$$\frac{S_n}{(1+r)^n},$$

i.e., the amount of money that should be deposited today to have exactly S_n dollars after n periods of time.

Financial streams

Suppose your enterprise wants to start a new project/investment. How can we decide whether is it convenient or not?

We need an estimate of the cash flows

$$\boldsymbol{x} = \{x_1, \dots, x_m\}$$

that the project will generate at known times $\{t_1 < \dots < t_m\}$.

- Any positive cash flow represents a profit that the company earns for deciding to undertake the project/investment.
- Any negative cash flow represents a loss that the company need to suffer for deciding to undertake the project/investment.

Financial streams

Example

GreenTech Innovations is considering a project to develop a high-efficiency solar panel with an initial investment of \$10 mln and annual costs of \$1 mln. The company does not expect any income in the first year. However, from the second year onwards, revenues are expected to start trickling in, with \$1 mln in the second year, increasing to \$3 mln in the third year, \$5 million in the fourth year, and reaching \$8 million by the fifth year.

The cash flows of the project (in mln of dollars) are thus

$$\begin{cases} x_0 &= &-10 & \Rightarrow \text{This is the initial cost.} \\ x_1 &= &-1 & \Rightarrow \text{No revenue yet, only costs.} \\ x_2 &= &+1-1=0 & \Rightarrow \text{Revenues minus costs.} \\ x_3 &= &+3-1=+2 & \\ x_4 &= &+5-1=+4 & \\ x_5 &= &+8-1=+7 & \end{cases}$$

Financial streams

We have thus

$$x_j = \underbrace{R_j}_{\text{Revenue}} - \underbrace{K_j}_{\text{Capital invested}}, \quad j = 0, 1, 2, 3, 4, 5.$$

Suppose **the project is not undertaken** and the capital is invested in the bank account. The maturity value is

$$K_0\,(1+r)^5 + K_1\,(1+r)^4 + K_2\,(1+r)^3 + K_3\,(1+r)^2 + K_4\,(1+r) + K_5$$

Suppose **the project is undertaken** and the revenues are invested in the bank account. The maturity value is

$$R_0\,(1+r)^5 + R_1\,(1+r)^4 + R_2\,(1+r)^3 + R_3\,(1+r)^2 + R_4\,(1+r) + R_5$$

If it occurs that

$$R_0\,(1+r)^5 + \cdots + R_4\,(1+r) + R_5 > K_0\,(1+r)^5 + \cdots + K_4\,(1+r) + K_5$$

it is convenient to undertake the project.

Financial streams

The decision rule

For a generic investment project, if

$$R_0\,(1+r)^m + \cdots + R_m > K_0\,(1+r)^m + \cdots + K_m, \qquad (\star)$$

it is more convenient to undertake the project than investing the capital in the bank account.

Re-arrange expression $(\star)$ to have

$$R_0 - K_0 + \frac{R_1 - K_1}{1+r} + \cdots + \frac{R_{m-1} - K_{m-1}}{(1+r)^{m-1}} + \frac{R_m - K_m}{(1+r)^m} > 0$$

Net Present Value and Present Value

The Net Present Value of the stream of cash flows:

$$\mathsf{NPV} \doteq \sum_{j=0}^{m} \frac{R_j - K_j}{(1+r)^j} = \sum_{j=0}^{m} \frac{x_j}{(1+r)^j}, \qquad \mathsf{PV} \doteq \sum_{j=1}^{m} \frac{x_j}{(1+r)^j}$$

Why the geometric series is so important

The geometric series

Recall that, for every number $x \in \mathbb{R}$, $x \neq 1$, we have

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x},$$

whence

$$\sum_{k=0}^{+\infty} x^k = \begin{cases} \frac{1}{1-x} & \Leftrightarrow |x| < 1 \\ +\infty & \Leftrightarrow x > 1 \\ \nexists & \Leftrightarrow x \leq -1 \end{cases}$$

In particular, if $|x| < 1$, we have

$$\sum_{k=1}^{+\infty} x^k = \sum_{k=0}^{+\infty} x^k - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

Geometric_series_template

The Net Present Value: special cases

- Annuity ($x_j = C$, $j = 1, \ldots, m$):

$$\mathsf{PV} = \sum_{j=1}^{m} \frac{C}{(1+r)^j} = C\left(\frac{1 - \frac{1}{(1+r)^{m+1}}}{1 - \frac{1}{1+r}} - 1\right) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^m}\right)$$

- Perpetuity ($x_j = C$, $j = 1, \ldots, \infty$):

$$\mathsf{PV} = \sum_{j=1}^{+\infty} \frac{C}{(1+r)^j} = C\left(\frac{1}{1 - \frac{1}{1+r}} - 1\right) = \frac{C}{r}$$

The Net Present Value: the Gordon formula

Suppose that the cash flow grows at a constant rate, that is

$$x_j = C\left(1+g\right)^j,$$

with $0 < g < r$, where r is the discount rate. Therefore $1+g < 1+r$ and $0 < (1+g)/(1+r) < 1$. We have

$$\mathsf{PV} = \sum_{j=1}^{m} \frac{x_j}{(1+r)^j} = C\sum_{j=1}^{m}\left(\frac{1+g}{1+r}\right)^j = C\left(\frac{1-\left(\frac{1+g}{1+r}\right)^{m+1}}{1-\frac{1+g}{1+r}} - 1\right).$$

when $m \to \infty$ we get

$$\mathsf{PV} = \sum_{j=1}^{+\infty} \frac{x_j}{(1+r)^j} = C\left(\frac{1}{1-\frac{1+g}{1+r}} - 1\right) = \frac{C\left(1+g\right)}{r-g}.$$

Financial streams

Exercise 1: GreenTech Innovations

GreenTech Innovations is considering a project to develop a high-efficiency solar panel with an initial investment of $10 mln and annual costs of $1 mln. The company does not expect any income in the first year. However, from the second year onwards, revenues are expected to start trickling in, with $1 mln in the second year, increasing to $3 mln in the third year, $5 million in the fourth year, and reaching $8 million by the fifth year.

`Cash_flows_GreenTech`

The Internal Rate of Return

Internal Rate of Return

Is the rate r which solves the equation:

$$NPV = 0$$

- The IRR is a property of the stream of cash flows.
- The definition does not guarantee the existence of a unique IRR. The equation $NPV = 0$ could have multiple solutions, or no solutions at all. The IRR makes sense only in situations in which the solution is unique.

Solving the break-even problem

We can use different tools:

1. Goal Seek,
2. Excel Solver.
3. IRR function (see next slide).

EXERCISE 2: COMPUTING THE BREAK-EVEN RATE OF A PROJECT

Consider the potential investment in a new manufacturing plant. The cost of the new plant is estimated to be \$1 mln in year 0. The average revenues from sales are estimated to be \$500,000 in the first year, \$100000 in the next 4 years, and then \$30,000 for the following 15 years. What is the break-even interest rate for the project?

We must find (provided that it exists) an r such that:

$$-1+\frac{0.5}{(1+r)}+\sum_{j=2}^{5}\frac{0.1}{(1+r)^j}+\sum_{j=6}^{20}\frac{0.03}{(1+r)^j}=0$$

To simplify, we have used millions of dollars as a unit of measure.

Break_even_problem

Exercise 3: Comparing compensation schemes

The construction company "Another Brick in the Wall" receives the offer to build a bridge. The bridge's construction requires three years, and the estimated yearly costs are \$2 mln, \$2.5 mln, and \$3 mln in the years 1, 2, and 3, respectively. Three compensation schemes are offered:

1. An advance compensation of \$6.5 mln, to be given at year 0.
2. A total compensation of \$7 mln split in two equal parts: the first before the start of the project, in year 0, and the second after a verification period for the bridge, in year 4.
3. A total compensation of \$7.5 mln split as follows: \$2.5 mln in year 0, \$1 mln in years 2, 3, and \$3 mln after bridge verification in year 4.

Which of the compensation schemes should the company prefer if r is below 5%? Which one if r is above 20%?

The_bridge_problem

The circularity problem in Excel

Exercise 4: Employee Bonus

Suppose your business makes a gross profit of \$500,000 and you want to calculate the net profit, i.e. after taxes and employee bonuses. The Federal Tax is 17%, the New York State Tax is 12%, and the employee bonus is 10%. However, while the federal and the state taxes are applied to the gross profit, the employee bonuses must be computed on your after-tax profits.

Compute the net profit and the bonus for employees.

This situation allows us to use circular references for iterative calculations in Excel.

The circularity problem in Excel

Profit before employee bonus and all taxed = \$500,000 (A)

Federal taxes = $\$500{,}000 \times 17\%$ (B)

New York State taxes = $\$500{,}000 \times 12\%$ (C)

Bonus for employees = (E) × 10% (D)

Profit after taxes and bonus = (A) – (B) – (C) – (D) (E)

`The_employee_bonus_problem`

WARNING

Enable iterative computation!