Practice session 3

Game Theory - MSc EEBL

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Exercise 1. ***Headphones war***

Two mobile phone producers, 1 and 2, use different standards for plugging headphones. Firm 1 is dominant and plays first. Firm 1 can either choose "no common standard" (N), a "basic common standard" (B) or a "sophisticated common standard" (S). If Firm A chooses N, the game ends. If Firm 1 chooses B or S then Firm B has to choose whether to "accept" (A) or "reject" (R) the standard. After Stage 2, the game ends. Payoffs are as follows:

- $(2,4)$ if Firm 1 chooses N;
- $(3,2)$ if Firm 1 proposes B and Firm 2 accepts (A);
- $(1,1)$ if Firm 1 proposes B and Firm 2 rejects (R);
- $(1,3)$ if Firm 1 proposes S and Firm 2 accepts (A);
- $(1,2)$ if Firm 1 proposes S and Firm 2 rejects (R).

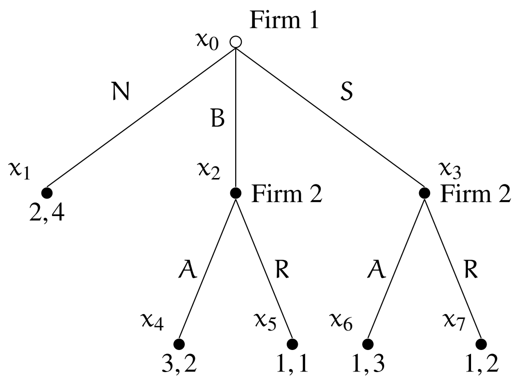
1. Write the game as an *extensive-form* game and the game tree.
2. Carefully write the strategy space of each player.
3. Find the *pure-strategy Nash equilibria* of the game.
4. Find the *subgame-perfect Nash equilibrium* of the game.
5. For each Nash equilibria which is not a SPNE, indicate what is the strategy that is not *sequentially rational* (noncredible threats).

Answer of Exercise 1.

1. Players are $N = \{1, 2\}$. Let us define nodes as follows (see the game tree below):

   - Set of nodes: $X = \{x_0, \ldots, x_7\}$.
   - Root: $r = x_0$.
   - Decision nodes: $X_1 = \{x_0\}$, $X_2 = \{x_2, x_3\}$.
   - Terminal nodes: $T = \{x_1, x_4, x_5, x_6, x_7\}$.
   - Action spaces: $A_{x_0} = \{N, B, S\}$, $A_{x_2} = A_{x_3} = \{A, R\}$.
   - Payoffs can be found in the game tree.

   The tree below corresponds to this game in extensive form.



2. For firm 1 the strategy space is the same as the action space, namely $S_1 = \{N, B, S\}$.

   For firm 2, however, its strategies must be contingent on what firm 1 does. We define its strategy space as follows: $S_2 = \{AA, AR, RA, RR\}$. In other words, firm 2's strategies must specify what to do after each observable action of firm 1. For instance, RA means that firm 2 will play R after B and A after S. Firm 2's strategies are only a couple of action (and not a triple) because firm 2 has to play only after firm 1 played B and S but not after N.

3. When looking for Nash equilibria, we can use the following payoff matrix:

		Firm 2			
		AA	AR	RA	RR
	N	(2,4)	(2,4)	(2,4)	(2,4)
Firm 1	B	(3,2)	(3,2)	(1,1)	(1,1)
	S	(1,3)	(1,2)	(1,3)	(1,2)

Using the standard way to find for Nash Equilibria we get:

		Firm 2			
		AA	AR	RA	RR
	N	$(2,\underline{4})$	$(2,\underline{4})$	$(\underline{2},\underline{4})$	$(\underline{2},\underline{4})$
Firm 1	B	$(\underline{3},\underline{2})$	$(\underline{3},\underline{2})$	$(1,1)$	$(1,1)$
	S	$(1,\underline{3})$	$(1,2)$	$(1,\underline{3})$	$(1,2)$

There are four (pure-strategy) Nash equilibria : (B, AA), (B, AR), (N, RA) and (N, RR).

4. Consider now SPNE, that is, players must be sequentially rational. For that matter, we need that firm 2's choice is a best-response to firm 1's action in every subgame. There are three subgames here: the whole game starting at x_0, and two others starting at x_2 and x_3, respectively.

   We will find the SPNE using *backward induction* reasoning. We first analyze what firm 2 does in subgames starting at x_2 and x_3 and then we can determine what firm 1 will do at x_0 given what firm 1 knows about firm 2's best responses.

   Consider first the one starting at x_2. It is clear that firm 2 must choose A after B because with A they get 2 instead of 1 with R.

   Consider now the one starting at x_3. Firm 2 must choose A after S because with A they get 3 instead of 2 with R.

   Therefore, in both of these subgames, firm 2 is better-off playing A than R (notice that then A is actually a dominant strategy for firm 2). This clearly shows that the Nash Equilibria (N, RA), (N, RR) and (B, AR) feature noncredible threats from firm 2 as it is not sequentially rational to play anything else that A for firm 2.

   Now, we know that firm 2 plays A both after B and R so that firm 1 can expect to receive 2 by playing N, 3 by playing B and 1 by playing S. Hence, firm 1 will choose to play B and the unique SPNE is (B, AA).

5. We know that firm 2's best response is to play A in each subgames it plays.

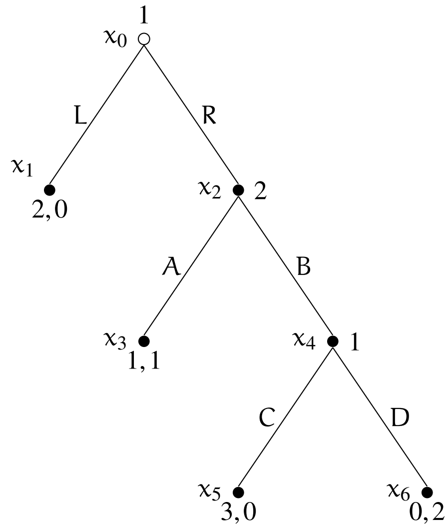
   Hence, the NE (B, AR) features a noncredible threat in the subgame starting at x_3.

   The NE (N, RA) features a noncredible threat in the subgame starting at x_2.

   And the NE (N, RR) features a noncredible threat in the subgame starting at x_2 and in the one starting at x_3.

Exercise 2. *A three-stage game*

Consider the following game.



1. Write the extensive form of this game.
2. Find all pure-strategy Nash equilibria.
3. Find the subgame-perfect Nash equilibrium.
4. Comment.

Answer of Exercise 2.

1. Relying on the game tree, the extensive form writes as follows:

   - Players are $N = \{1, 2\}$.
   - Set of nodes: $X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$.
   - Root: $r = x_0$.
   - Decision nodes: $X_1 = \{x_0, x_4\}$, $X_2 = \{x_2\}$.
   - Terminal nodes: $T = \{x_1, x_3, x_5, x_6\}$.
   - Action spaces: $A_{x_0} = \{L, R\}$, $A_{x_2} = \{A, B\}$, $A_{x_4} = \{C, D\}$.
   - Payoffs can be found in the game tree.

   We have to be careful here, player 1 may have to play two times if they first chose R and then player 2 chose B.

2. Let us first define the strategy space of each player. For player 1 we have $S_1 = \{LC, LD, RC, RD\}$ and for player 2 $S_1 = \{A, B\}$. We can write the payoff matrix to find Nash equilibria of this game.

		2	
		A	B
	LC	2,0	2,0
	LD	2,0	2,0
1	RC	1,1	3,0
	RD	1,1	0,2

We can underline each player's best responses as follows:

		2	
		A	B
	LC	$\underline{2},\underline{0}$	$2,\underline{0}$
	LD	$\underline{2},\underline{0}$	$2,\underline{0}$
1	RC	$1,\underline{1}$	$\underline{3},0$
	RD	$1,1$	$0,\underline{2}$

Hence, we have two pure-strategy Nash equilibria, namely (LC, A) and (LD, A).

3. We have three subgames: The game itself starting at x_0, a second one starting at x_2 and a third one starting at x_4.

   To find the SPNE, we can proceed by backward induction. Starting at the end of the game, at x_4, we find that C is player 1's Nash equilibrium strategy of this subgame.

   Going backward to x_2, player 2 has the choice between playing A and get 1 or playing B and get 0 (because player 1 will play C for sure after B). Hence, it is better to play A than B for player 2.

   Finally, starting at x_0, player 1 can either play L and get 2 or play R and get 1 (as after R player 2 will play A for sure). Hence, player 1 will choose L.

   It follows that the only SPNE of this game is (LC, A).

   Note: We have found that (LC, A) and (LD, A) were NE of the whole game. It is clear that C is the NE of the subgame starting at x_4.

   We can also check that (C, A) is a NE of the subgame starting at x_2. Indeed, this subgame can be written using the following payoff matrix:

		2	
		A	B
1	C	1,1	3,0
	D	1,1	0,2

   Underlying best responses we obtain:

		2: A	2: B
1	C	$\underline{1},\underline{1}$	$\underline{3},0$
1	D	$\underline{1},1$	$0,\underline{2}$

Hence, (C, A) is a NE of the subgame starting at x_2.

As all players' strategies in (LC, A) are a NE in each of their corresponding subgames, then it is a SPNE.

4. The other Nash equilibrium, (LD, A) is not SPNE as D is not a sequentially rational move at x_4.

   We have to be careful here because both (LC, A) and (LD, A) result in the same *outcome of the game*, that is, in both cases the game ends at x_1 after player 1 chose L. But there are distinct objects according to our theory because what matters is not only what happens in the end but also the players' strategies at every possible subgame, even though some are not played.

Exercise 3. *Stackelberg Duopoly*

Two firms compete in a market by setting the quantities of a (homogeneous) good to produce (q_i, q_j). Each firm faces a *constant marginal cost* $c_k \in \mathbb{R}_+$. The two firms face the *inverse demand function* $P(Q) = a - bQ$, where Q is the aggregate quantity produced and $(a, b) \in \mathbb{R}^2_+$ are demand parameters. Payoffs are given by each firm's profits.

The difference with the Cournot setting is that we are not going to assume that firms choose their quantity simultaneously. Instead, we are going to assume that one firm, say i, is the *leader* while firm j is the *follower*. The leader, firm i, sets its quantity q_i first. Then the follower, firm j, observes q_i and decides its own quantity q_j. This is what we call a Stackelberg Duopoly.

The game is a dynamic game in which firm i plays first a $q_i \in \mathbb{R}_+$ and second firm j plays a $q_j \in \mathbb{R}_+$. Then, the two quantities determine the market price through the inverse demand function and firms receives their profits.

1. Write the profit of each firm at the end of the game, for any given pair of quantities (q_i, q_j).

2. Starting at the second period, assume that firm i's choice of quantity in the first period is q_i. Find the best-response function of firm j to this quantity q_i. Call this best-response function $q_j(q_i)$.

3. Starting now at the first period, find firm i's optimal choice of q_i given that it anticipates firm j to best respond according to $q_j(q_i)$. Call the SPNE of this game (q_i^*, q_j^*).

Answer of Exercise 3.

1. Profits are given by the same exact expressions as in the Cournot case. Given a pair of quantities (q_i, q_j), the write as follows:

$$\pi_i(q_i, q_j) = (a - b(q_i + q_j) - c_i)q_i,$$
$$\pi_j(q_i, q_j) = (a - b(q_i + q_j) - c_j)q_j.$$

2. Assume firm i has chosen to produce q_i in the first period. Firm j observes this quantity and choose its own q_j to maximize its profits, that is, for any given q_i

$$\max_{q_j} \pi_j(q_i, q_j) = (a - b(q_i + q_j) - c_j)q_j.$$

   Notice that this problem is completely equivalent (mathematically speaking) to that of firm j in the Cournot setting. We should therefore expect firm j's best response function to be the same as in the Cournot setting.

   Indeed, the first-order condition of this problem writes

$$\frac{\partial \pi_j(q_i, q_j)}{\partial q_j} = 0 \implies a - 2bq_j - bq_i - c_j = 0 \implies q_j(q_i) = \frac{a - c_j - bq_i}{2b}.$$

   The best response function of firm j is the same as in the Cournot setting.

3. Moving to the first period of this game, firm i anticipates that firm j will set q_j according to $q_j(q_i)$. Hence, firm i will take this dependency into account into its profit function. Firm i chooses q_i to solve

$$\max_{q_i} \pi_i(q_i, q_j(q_i)) = \big(a - b\big(q_i + q_j(q_i)\big) - c_i\big)q_i.$$

   The important difference with the Cournot setting lies in firm i's profit function. The fact that firm i takes the best-response function of firm j explicitly in its maximization problem accounts for the dynamic aspect of this game.

   Firm i's profit can also be explicitly written as follows:

$$\max_{q_i} \pi_i(q_i, q_j(q_i)) = \big(a - b\big(q_i + \frac{a - c_j - bq_i}{2b}\big) - c_i\big)q_i.$$

   Or even more simplified

$$\max_{q_i} \pi_i(q_i, q_j(q_i)) = \big(a - b\big(\frac{1}{2}q_i + \frac{a - c_j}{2b}\big) - c_i\big)q_i.$$

The first-order condition of this problem writes

$$a - b\left(\frac{1}{2}q_i + \frac{a - c_j}{2b}\right) - c_i - \frac{1}{2}bq_i = 0.$$

Solving for q_i we obtain the SPNE quantity q_i^*:

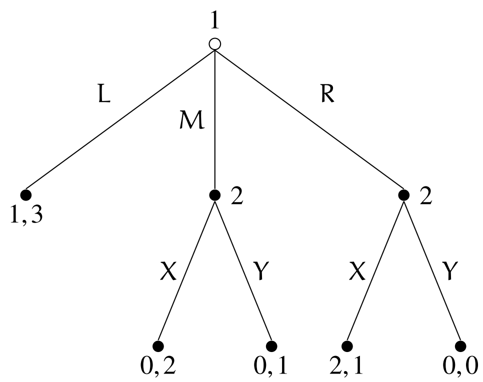
$$q_i^* = \frac{a + c_j - 2c_i}{2b}.$$

To determine the SPNE quantity q_j^*, we simply have to plug q_i^* into firm j's best-response function. We obtain

$$\begin{aligned} q_j^* = q_j(q_i^*) &= \frac{a - c_j - bq_i^*}{2b} \\ &= \frac{a - c_j}{2b} - \frac{1}{2}\frac{a + c_j - 2c_i}{2b} \\ &= \frac{a + 2c_i - 3c_j}{4b}. \end{aligned}$$

Exercise 4. *Additional exercise*

Consider the following dynamic game between player 1 and player 2.



1. Write this game as a payoff matrix. Carefully explain why you write player 2's strategies as a couple of actions.

2. Find all *pure-strategy Nash equilibria* of this game.

3. Find the unique *subgame-perfect Nash equilibrium*.

4. Explain why (L, XY) is a Nash equilibrium of the dynamic game but not a subgame-perfect Nash equilibrium.

Answer of Exercise 4.

1. Player 2's action space is $A_2 = \{X, Y\}$, however, player 2 has to choose what to play after M and after R. Hence, player 2's strategy space can be written as $S_2 = \{XX, XY, YX, YY\}$.

   We can now construct the payoff matrix of the normal-form game:

		2			
		XX	XY	YX	YY
	L	$1,3$	$1,3$	$1,3$	$1,3$
1	M	$0,2$	$0,2$	$0,1$	$0,1$
	R	$2,1$	$0,0$	$2,1$	$0,0$

2. We identify best responses in the matrix as follows.

		2			
		XX	XY	YX	YY
	L	$1,\underline{3}$	$\underline{1},\underline{3}$	$1,\underline{3}$	$\underline{1},\underline{3}$
1	M	$0,\underline{2}$	$0,\underline{2}$	$0,1$	$0,1$
	R	$\underline{2},\underline{1}$	$0,0$	$\underline{2},\underline{1}$	$0,0$

   Hence we have four Nash equilibria: (L, XY), (L, YY), (R, XX) and (R, YX).

3. Here we can easily notice that X is a strictly dominant strategy for player 2. Playing X is indeed strictly better than playing Y both after M ($2 > 1$) and R ($1 > 0$). Hence, X is what player 2 will play in both subgames.

   Foreseeing this behavior, player 1 can either play L and get 1, play M and get 0, or play R and get 2. It is straightforward that player 1 will therefore choose to play R.

   The unique SPNE of this game is (R, XX).

4. (L, XY) is a Nash equilibrium because the strategy XY acts as a threat to player 1. Indeed, player 1 expects to get 0 if they play either M or R. Hence, it is a best response to play L in that case. Conversely, when player 1 plays L, it is a best response to play XY for player 2 as it ensures them their highest possible payoff (3).

   However, this logic works only because the way we solve for the Nash equilibrium ignores the dynamic feature of the game. As we have seen in class, it seems unreasonable to assume that in case player 1 would deviate to M or R, player 2 would indeed stick to XY as it would be preferable for them to choose XX. In other words, playing Y after R is not sequentially rational, or, in that specific case it is a noncredible threat. Hence, (L, XY) is a Nash equilibrium but not a SPNE as it features strategies that are not sequentially rational.