Practice session 5

Game Theory - MSc EEBL

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Exercise 1. *A simple static game of incomplete information*

In this exercise, the goal is to determine whether some strategy profiles are Bayesian Nash equilibria or not. See Exercise 2 for the method to find *all* Bayesian Nash equilibria in a similar game.

Consider the following *static game of incomplete information.*

	L	R
U	2,2	1,−3
D	−3,1	0,0

prob. 1/2

	L	R
U	0,0	1,−3
D	−3,1	2,2

prob. 1/2

We further assume that player 1 is fully informed about which of these two games is played whereas player 2 only knows that each game has the same probability of occurring.

1. Write the normal-form representation of this game.

2. For this question only, assume that player 1 **always** play U, that is, player 1's strategy is UU.

   (a) What is player 2's expected payoff when playing L? When playing R? Deduce player 2's best response to UU.

   (b) Show that if player 2 plays L, then UU is actually a best response for player 1.

   (c) Conclude whether (UU, L) is a *Bayesian Nash equilibrium.*

3. Show that (UD, R) is not a Bayesian Nash equilibrium of this game.

4. Assume now that player 2 believes that the game on the left is played with probability $\alpha \in [0,1]$.

(a) Find a threshold value $\hat{\alpha}$, such that for all $\alpha \leqslant \hat{\alpha}$, R is a best response to UD for player 2.

(b) Is (UD, R) is a Bayesian Nash equilibrium for $\alpha \leqslant \hat{\alpha}$?

Answer of Exercise 1.

1. For players and actions it is very simple: $N = \{1, 2\}$, $A_1 = \{U, D\}$ and $A_2 = \{L, R\}$.

   In order to define players' type spaces, we have to interpret the sentence about players' information. We can define type spaces such that player 1 has two types (one for each game) and player 2 only one type. For instance, let $T_1 = \{l, r\}$ and $T_2 = \{t_2\}$ where l and r correspond to the game on the left and the game on the right, respectively. That way, it is clear that player 1 is fully informed about which game is played as they know their own type and have no uncertainty about player 2's type. And player 2 is uncertain whether player 1 is of type l or r, that is, which game is played.

   We should also defined players' beliefs. Clearly $p_1(t_2) = 1$ and $p_2(x) = p_2(y) = 1/2$.

   Payoffs are defined in the matrices where they represent each player's utility depending on the joint actions and on the joint types. For instance $u_1(U, L; l, t_2) = 2$, $u_1(U, L; r, t_2) = 0$, $u_2(U, L; r, t_2) = 2$, $u_2(D, R; r, t_2) = 2$.

2. (a) When player 1 plays U and player 2 plays L the expected payoff of player 2 is:

$$\frac{1}{2}2 + \frac{1}{2}0 = 1.$$

   When player 1 plays U and player 2 plays R the expected payoff of player 2 is:

$$\frac{1}{2}(-3) + \frac{1}{2}(-3) = -3.$$

   So player 2 prefers to play L without any ambiguity.

(b) For player 1, it is straightforward that playing U is a best response in the left game as it is a dominant strategy for this player. In the right game, when player 2 plays L, player 1 prefers U to D as it gives them 0 instead of -3.

(c) As UU is a best response to L and L is a best response to UU, then (UU, L) is a Bayesian Nash equilibrium.

3. To prove that (UD, R) is not a BNE of this game, we have to prove that for at least one player, there is a possible profitable deviation.

Let us assume that player 1 chooses UD and let us compute player 2's expected payoff when playing R:

$$\frac{1}{2}(-3) + \frac{1}{2}2 = -\frac{1}{2}.$$

Player 2 could instead *deviate* and play L and they would get

$$\frac{1}{2}2 + \frac{1}{2}1 = \frac{3}{2}.$$

It is clear that for player 2, R is not a best response to UD. Hence, $(\mathsf{UD}, \mathsf{R})$ is not a BNE of this game.

(a) We want player 2 to prefer R to L against UD.

Before doing any computation we can try to guess whether we expect this to happen for low or high values of α. Notice that when player 2 plays R against UD it results a payoff of -3 when it's the game on the left and a payoff of 2 when it's the on the right. Hence, when playing R player 2 strongly prefer the game on the right. The opposite is true for playing L against UD as player 2 gets 2 if the game is the one on the left and 1 if it's that on the right. Therefore, the less likely is the game on the left, the better it is to play R and the worse it is to play L against UD. We therefore expect that R will be a best response to UD for low values of α.

Let us now solve this problem formally. First, let us compute the expected payoff of player 2 when playing R against UD for any possible α:

$$\alpha(-3) + (1 - \alpha)2 = 2 - 5\alpha.$$

Now let us compute player 2's expected payoff when playing L against UD for any possible α:

$$\alpha 2 + (1 - \alpha)1 = 1 + \alpha.$$

Hence, player 2 prefers R to L against UD whenever

$$2 - 5\alpha \geqslant 1 + \alpha.$$

Solving this inequality for α we immediately find that it is true for $\alpha \leqslant \frac{1}{6}$.

We can define $\hat{\alpha} = \frac{1}{6}$ and so we have that R is a best response to UD whenever $\alpha \leqslant \hat{\alpha}$.

(b) We know that for $\alpha \leqslant \hat{\alpha}$, R is a best response to UD. Then we have to check whether UD is a best response to R. As previously said, U is a dominant strategy in the game on the left and so we know that player 1 cannot play anything else. It remains to

check whether D is a best response to R in the game on the right for player 1. Indeed, in the game on the right, player 1 prefers D to U against R ($2 > 1$).

We have that both UD is a best response to R and R is a best response to UD whenever $\alpha \leqslant \hat{\alpha}$. We can conclude that $(\mathsf{UD}, \mathsf{R})$ is a Bayesian Nash equilibrium of the game whenever $\alpha \leqslant \hat{\alpha}$.

Exercise 2. *Hiring Decisions*

Consider a Firm (F) and a Worker (W). If W has a *high ability* ($T_w = \text{high}$), they would like to *Work* when they are hired; if instead W has a *low ability* ($T_w = \text{low}$), they would rather *Shirk*.

F wants to *Hire* W if the latter is willing to *Work* and *Not Hire* them otherwise. W knows their ability level while F does not. However, F *believes* that W's ability is high with probability $\mathbb{P}(T_w = \text{high}) = \frac{2}{3}$ and low with probability $\mathbb{P}(T_w = \text{low}) = \frac{1}{3}$. The payoffs (different for each type of W) are as follows:

	Work	Shirk
Hire	1,2	0,1
Don't	0,0	0,0

$T_w = \text{High}$

	Work	Shirk
Hire	1,1	−1,2
Don't	0,0	0,0

$T_w = \text{Low}$

1. Describe the game as a normal-form game.

2. Write the strategy profile for each player.

3. Write the game as an extensive-form game.

4. Find all (pure-strategy) Bayesian Nash equilibria of the game.
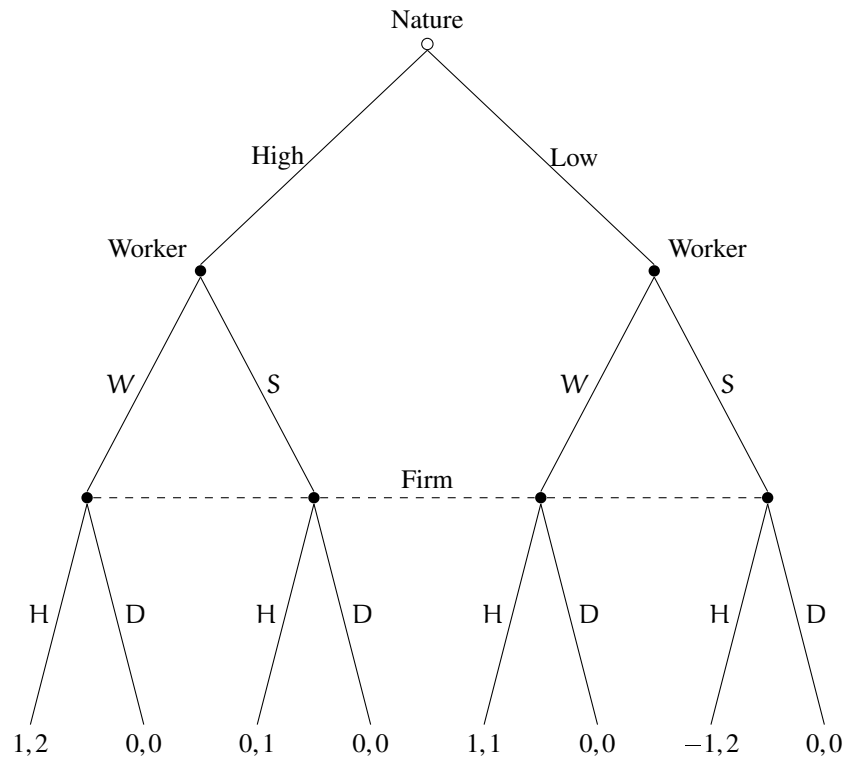
Answer of Exercise 2.

1. The normal-form writes as follows

   - Players: $N = \{F, W\}$.
   - Action sets: $A_F = \{H, D\}$, $A_W = \{W, S\}$.
   - Type spaces: $T_F = \{t_F\}$, $T_W = \{t_{W1}, t_{W2}\} = \{\text{High}, \text{Low}\}$.
   - Beliefs: $p_F(t_{W1}) = 2/3$, $p_W(t_F) = 1$.
   - Payoffs: Given by the two matrices.

2. The firm has to choose either to Hire (H) or Don't hire (D).

   The worker has to choose an action for each type: Work (W) or Shirk (S) for each element of T_W.

   We denote the worker's strategy profile by WW, WS, SW and SS where for instance WS means that the worker of type $t_{W1} =$ High choose to Work and the worker of type $t_{W2} =$ Low choose to Shirk.

3. The extensive form is shown below.

4. Let us find the best-response, for each player and each type. We first need to compute the interim expected payoff of the Firm.

   Let $\mu_F(a_F, a_{W1} a_{W2})$ denote the interim expected payoff of the firm when it chooses $a_F \in A_F$, a worker of type t_{W1} chooses $a_{W1} \in A_W$, and a worker of type t_{W2} chooses $a_{W2} \in A_W$.

When the firm chooses to hire it expects:

$$\mu_F(H, WW) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 1 = 1,$$
$$\mu_F(H, WS) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = \frac{1}{3},$$
$$\mu_F(H, SW) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{1}{3},$$
$$\mu_F(H, SS) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot (-1) = -\frac{1}{3}.$$

When the firm chooses *not* to hire, notice that the firm's ex post payoff is 0 regardless of the worker's choice. Then it is easy to see that $\mu_F(D, a_{W1} a_{W2}) = 0$ for every $a_{W1}, a_{W2} \in A_W$.

It is now possible to create a new payoff matrix as follows:

F \W	WW	WS	SW	SS
H	1 ; (2,1)	$\frac{1}{3}$; (2,2)	$\frac{1}{3}$; (1,1)	$-\frac{1}{3}$; (1,2)
D	0 ; (0,0)	0 ; (0,0)	0 ; (0,0)	0 ; (0,0)

Investigation of this matrix reveals that the firm's best response is as follows:

- H when WW, WS and SW;
- D when SS.

When the worker is of type High:

- W when H;
- W or S when D.

When the worker is of type Low:

- S when H;
- W or S when D.

It is possible to represent best responses (in red) in the payoff matrix as follows:

F \W	WW	WS	SW	SS
H	1 ; (2,1)	$\frac{1}{3}$; (2,2)	$\frac{1}{3}$; (1,1)	$-\frac{1}{3}$; (1, 2)
D	0 ; (0,0)	0 ; (0,0)	0 ; (0,0)	0 ; (0,0)

There are two pure-strategy BNE: (H, WS) and (D, SS).

If you want to convince yourself, take for instance (H, WS) and assume that one player is given the chance to change their action (even only one type).

The firm could choose to play D instead of H but then it would receive 0 instead of 1/3. The firm clearly does not want to change its action.

Consider the High-type worker now. They could choose to S rather than W. In that case, they would receive 1 instead of 2. It is then not profitable to deviate.

Finally, consider the Low-type worker. They could choose to W rather than S. In that case, they would receive 1 instead of 2. It is then not profitable to deviate.

Then it is clear that at (H, WS), no one would like to deviate.

Exercise 3. *Matching technologies*

Consider two firms: Firm 1 is producing a video game console and firm 2 is producing a video game. Each firm can choose to *cooperate* (C) or *not cooperate* (N). For instance, firm 1 can cooperate by providing better development tools and firm 2 can cooperate by better optimizing their game to run on firm 1's console.

Each firm's product can be either of type *High* (H) or *Low* (L). Payoffs are as follows.

	C	N
C	3,3	−2,0
N	0,−2	0,0

$G_1 : \{H, H\}$

	C	N
C	−1,−1	−2,0
N	0,−2	0,0

$G_2 : \{L, L\}$ or $\{H, L\}$ or $\{L, H\}$

In words, firms enjoy cooperation only when they both have high types. Each firm only knows their own type. They both think that the probability that the other firm is of type H is 1/2. Let $p_1(H) = p_2(H) = 1/2$ denote this belief.

1. (a) What is the probability that firms are in configuration G_1? Same question for G_2.

   (b) Now assume firm 1 knows it is of type H. From its point of view, what is the probability that it faces configuration G_1?

2. What is a strategy for firm $i = 1, 2$ in this game?

3. Assume firm 2 decides to cooperate for each of its type, that is, firm 2 plays CN.

   (a) What is the best response of firm 1 when it is of type H?

   (b) What is the best response of firm 1 when it is of type L?

   (c) Deduce a Bayesian Nash equilibrium of this game.

Answer of Exercise 3.

1. (a) Configuration G_1 occurs only when both firms have a high type, that is, $p_1(H)p_2(H) = 1/4$. On the other hand, G_2 occurs for $\{L, L\}$, $\{H, L\}$ or $\{L, H\}$ which corresponds to probability $p_1(L)p_2(L) + p_1(H)p_2(L) + p_1(L)p_2(H) = 3/4$.

   (b) Here we want to evaluate the probability of the event $\{H, H\}$ given that firm 1 knows that is is of type H itself. Let us denote by $p_1(\{H, H\}|H)$ this probability. Applying Bayes' rule we have

$$p_1(\{H, H\}|H) = \frac{p_1(\{H, H\})}{p_1(H)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

   We can therefore deduce that if firm 1 is of type H it knows that G_2 occurs with probability $1/2$.

   Is is also straightforward that if firm 1 knows that it is of type L then it knows for sure that it faces configuration G_2.

   By symmetry, the same applies to firm 2.

2. Each firm has to formulate a choice C or N for each of their type. Hence, firm i's possible strategies in this game can be written CC, CN, NC and NN where the first letter refers to the action chosen when firm i is of type H and the second when it is of type L.

3. (a) Assume firm 2 plays CN and firm 1 is of type H. It follows that firm 1 knows that configuration G_1 and G_2 are equally likely to occur. Firm 1 anticipates that in configuration G_1, firm 2 will play C as it can only be of type H. In configuration G_2, firm 1 anticipates that firm 2 will play N as being in G_2 must be because we have $\{H, L\}$, i.e., firm 2 is L.

   We can compute it's expected payoff when playing C:

$$\frac{1}{2}3 + \frac{1}{2}(-2) = \frac{1}{2},$$

   and when playing N

$$\frac{1}{2}0 + \frac{1}{2}0 = 0.$$

   Clearly, firm 1 of type H prefers to cooperate (C).

   (b) Still assume firm 2 plays CN and but firm 1 is now of type L. Firm 1 now knows for sure that it faces configuration G_2. However, we must be careful because firm 1 is still uncertain about whether we are in G_2 because we have $\{L, H\}$ or because we have $\{L, L\}$. Types being independent, the probability that firm 2 is of type H is $1/2$.

Hence the expected payoff of firm 1 of type L playing C against CN writes:

$$\frac{1}{2}(-1) + \frac{1}{2}(-2) = -\frac{3}{2},$$

and when playing N

$$\frac{1}{2}0 + \frac{1}{2}0 = 0.$$

Hence firm 1 of type L prefers not to cooperate (N).

(c) We have shown that when firm 2 plays CN, firm 1's best response is CN. As the game is symmetric, it means that we can also say that when firm 1 plays CN, firm 2's best response is CN.

It immediately follows that (CN, CN) is a Bayesian Nash equilibrium of this game.