Practice session 6

Game Theory - MSc EEBL

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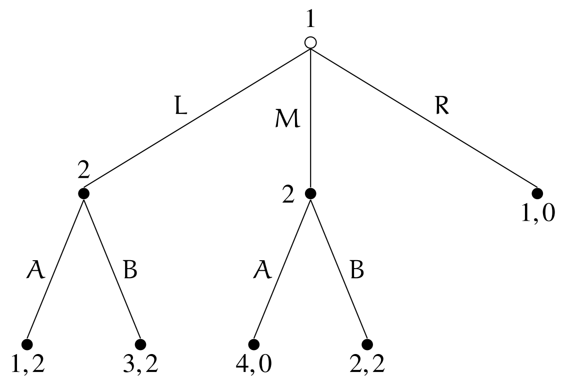
Exercise 1.

Consider the following **dynamic game of complete information**.



1. Compute the *Nash Equilibria* and the *Subgame-Perfect Nash Equilibria*.
2. Are the sets of NE and SPNE different? The same? Carefully explain why it is the case.

From now on, assume that **player 2 cannot distinguish** action L from action M.

3. Explain how we should modify the tree above to represent this game.
4. The *Nash Equilibrium* of this game is (L, B). Explain why the strategy of player 2 is not a couple of actions (for instance AB or BB) like it was the case in question 1.
5. Consider the strategy profile (L, B). Is it a *Perfect Bayesian Nash Equilibrium*?
6. Show that (M, B) **is not** a *Perfect Bayesian Nash Equilibrium*.

Answer of Exercise 1.

1. Player 1's strategies are L, M and R while player 2's strategies are AA, AB, BA and BB. The Nash Equilibria are obtained by solving the following matrix-form game.

	AA	AB	BA	BB
L	$1,\underline{2}$	$1,\underline{2}$	$3,\underline{2}$	$\underline{3},\underline{2}$
M	$\underline{4},0$	$\underline{2},\underline{2}$	$\underline{4},0$	$2,\underline{2}$
R	$1,\underline{0}$	$1,\underline{0}$	$1,\underline{0}$	$1,\underline{0}$

   Hence, (M, AB) and (L, BB) are Nash equilibria.

   For SPNE, it is clear that after L player 2 is indifferent between A and B. After M, player 2 prefers B. Both NE feature **credible threats**. AB and BB are the possible best-response strategies.

   Then (M, AB) is a SPNE because B is a best response to M and A is a credible threat to L. Player 1 therefore prefers to play M and get 2 than playing L and get 1.

   Finally, (L, BB) is also a SPNE as BB is also a best response and players prefers to play L to get 3 than playing M and get 2.

2. The sets of NE and SPNE are the same because player 2 is indifferent between A and B after L and this allows them to credibly threat player 1 to play A after L to avoid a deviation of player 1 from M to L.

3. Draw a dashed line at the two decision nodes of player 2.

4. First, let us compute the NE in this game.

	A	B
L	$1,\underline{2}$	$\underline{3},\underline{2}$
M	$\underline{4},0$	$2,\underline{2}$
R	$1,\underline{0}$	$1,\underline{0}$

   Unique pure-strategy NE, (L, B).

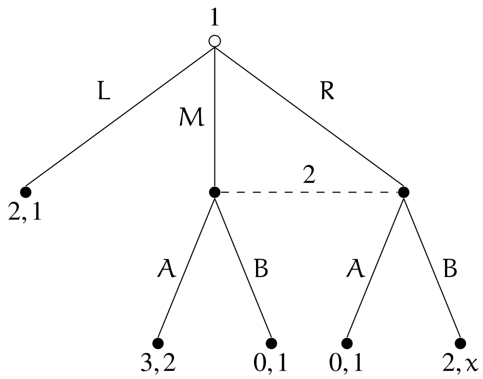
   Player 2 does not observe the action taken by player 1 so that their strategy space is the same as their action space as player 2 has no way to form different strategies after different histories.

5. Consider (L, B). It induces beliefs $\mu = 1$, where μ represents player 2's belief that player 1 has chosen L. Action B is a best response to L and in that case player 1 gets 3. If player 1 deviates to M, player 2 will stick to B (as player 2 wrongly believes that player 1 has played L) and player 1 gets $2 < 3$. Finally if player 1 deviates to play R they get $1 < 3$. Hence, (L, B) and $\mu = 1$ is a PBNE.

6. Consider (M, B). Player 2's belief must then be $\mu = 0$. Action B is a best response to M and in that case player 1 gets 2. If player 1 deviates to L, player 2 will stick to B (as player 2 wrongly believes that player 1 has played M) and player 1 gets $3 > 2$. Hence, player 1 has an incentive to deviate. It follows that (M, B) is not a PNBE.

Exercise 2. *Perfect Bayesian Nash Equilibrium*

Consider the following dynamic game of incomplete information.



Assume that x is either equal to 0 or to 3. Let $\mu \in [0, 1]$ and $1 - \mu$ denote player 2's beliefs about M and R, respectively.

1. Assume for now that $x = 0$.

   (a) Can you identify a dominant strategy for player 2?

   (b) If player 1 had to choose only between M and R, what would they do?

   (c) Deduce what should be player 2's beliefs about M and R.

   (d) Find a perfect Bayesian Nash equilibrium of this game.

2. Assume for now that $x = 3$.

   (a) For which belief $\mu \in [0, 1]$, player 2 prefers to play A to B?

   (b) Characterize perfect Bayesian Nash equilibria with respect to the value of μ.

Answer of Exercise 2.

1. (a) For player 2, it is clear that A is dominant strategy when $x = 0$.

   (b) Player 1 expects player 2 to choose A is they play M or R. Clearly playing M is better than playing R ($1 > 0$).

   (c) Player 2 can guess that if the game reaches the information set at which they play, player 1 must have chosen M. Hence, player 2's belief when the game reaches the information set is $\mu = 1$.

   (d) We already know that player 1 prefers to play M rather than R. It is also true that player 1 prefers to play M rather than L ($3 > 2$).

   Hence, a perfect Bayesian Nash equilibrium of this game is (M, A) and $\mu = 1$.

2. (a) We have to compute player 2's expected payoff under belief μ. First, consider player 2's expected payoff when playing A:

$$\mu 2 + (1 - \mu)1 = 1 + \mu,$$

   and when playing B

$$\mu 1 + (1 - \mu)3 = 3 - 2\mu.$$

   Hence player 2's prefers to play A whenever

$$1 + \mu \geqslant 3 - 2\mu \Leftrightarrow \mu \geqslant \frac{2}{3}.$$

   And conversely, player 2 prefers to play B whenever $\mu \leqslant \frac{2}{3}$.

   (b) Let us start with the case $\mu \geqslant \frac{2}{3}$. In that case, player 2 will play A if player 1 plays either M or R. The same reasoning as in question 1 applies: Player 1 strictly prefers M to L ($3 > 2$) and to R ($3 > 0$).

   Hence we have that (M, A) and $\mu \geqslant \frac{2}{3}$ is a PBNE when $x = 3$.

   Now let us assume that $\mu \leqslant \frac{2}{3}$. Then player 2 prefers B to A. This means that player 1 is not anymore interested in playing M as they would get 0. Player 1 is now indifferent between L and R as they both give them a payoff of 2.

   Therefore, we have that both (L, B) and $\mu \leqslant \frac{2}{3}$, and (R, B) and $\mu \leqslant \frac{2}{3}$ are PBNE of this game when $x = 3$.

Exercise 3. ***Rational Handicap*** (*Zahavi, 1975 and Grafen, 1990*)

Male peacocks grow beautiful and colorful tails. It is, however, a handicap as it makes the male more noticeable to predators. From the Darwinian evolutionary theory, it may seem surprising that male peacocks with the most noticeable tails survive the law of evolution and this question has been puzzling biologists for a long time.

One of the possible answer to this problem is to consider that peacocks are playing the following signaling game. Nature chooses the male peacock type: With probability p the male is "strong" (H) and with probability $1-p$ it is "weak" (L). The male privately observes his type and **chooses** whether to "grow a colorful tail" (T) or "not" (nT). Then, a female peacock observes the male's choice (but not his type) and must decide to "mate" (M) or not (nM).

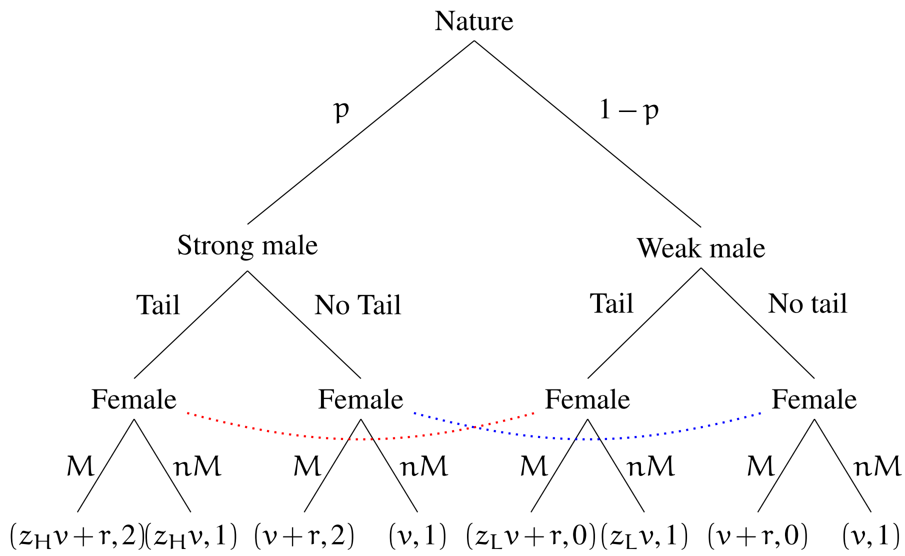
The female's payoffs are as follows: Mating with a type H yields 2, Mating with a type L yields 0 and not mating yields 1. The male's payoff are as follows: Not growing a colorful tail yields $v > 0$, growing a colorful tail when H yields $z_H v$ and growing a colorful tail when L yields $z_L v$. We assume $z_L < z_H < 1$. Finally, a male gets an additional payoff $r > 0$ if the female decides to mate, irrespective of his type or tail choice.

1. Write the normal-form representation and represent the game in a tree.
2. Find the pooling equilibria.
3. Find the separating equilibria.
4. Discuss the term "rational handicap".

Answer of Exercise 3.

1. Normal-form:

    - Players: $N = \{\text{Male}, \text{Female}\}$
    - Action spaces: $A_M = \{T, nT\}$, $A_F = \{M, nM\}$
    - Types: $T_M = \{H, L\}$, $T_F = \{t_F\}$
    - Prior beliefs: $p_M(t_F) = 1$, $p_F(H) = p$
    - Payoffs and information sets: In the tree



2. Let us first consider pooling equilibria (which are PBE). By definition, a pooling equilibrium is such that different types of the sender choose the same action so that their message is *not revealing*.

   Here, the sender is the male peacock and there are two possible pooling equilibria: One in which both types (H and L) grow a tail and one in which both type do not grow a tail.

   Let $\mu_T = \mathbb{P}(H \mid T)$ and $\mu_{nT} = \mathbb{P}(H \mid nT)$ denote the female's posterior belief about the male's type given what she observes. Let us now consider each pooling equilibrium separately.

   ■ **(P_{TT}) Both types grow a tail.** If this is an PBE, the female's posterior beliefs must then be $\mu_T = p$ and $\mu_{nT} \in [0,1]$.

   Indeed, it is clear that the information set that occurs after observing T is on the equilibrium path (occurs with positive probability, namely 1 here). Therefore, one can use Bayes' rule to compute the posterior belief μ_T given that for this strategy profile $\mathbb{P}(T \mid H) = \mathbb{P}(T \mid L) = 1$. The female learns nothing from the male's action and thus only use the exogenous information that a strong male occurs with probability p.

   On the contrary, the information set that occurs after observing nT is off the equilibrium path and μ_{nT} cannot be determined by Bayes' rule here. It must then remain undetermined.

   Given those posterior beliefs, assume that the female does not mate with a male who grows a tail. Then, if the male plays the pooling strategy " Both types grow a tail", type H gets $z_H v$ and type L gets $z_L v$. If, however, each type decides not to grow a tail each gets $pv + (1-p)v = v$. As $z_L < z_H < 1$ it is then clear that both males would prefer to deviate from T if the female chose not to mate with a male with a tail. As a result, if this pooling equilibrium exists, it is necessary that the female chooses to mate when she observes a tail.

Given her beliefs, if the female mates after observing a tail she gets $\mu_T 2 + (1-\mu_T)0 = 2p$ and if she does not mate she gets 1. A first necessary condition for the female to mate is then

$$2p \geqslant 1 \Leftrightarrow p \geqslant \frac{1}{2},$$

that is, it must be more likely that the male is H than L. We now assume $p \geqslant \frac{1}{2}$ for the rest of this part (otherwise it is for sure not an equilibrium).

Then it must also be a best response for both type of males to grow a tail given the female's posterior beliefs and that she chooses to mate after observing a tail. It is straightforward that if the female chooses to mate even after nT both males would deviate: Thus it is necessary that the female prefers not to mate after nT which occurs whenever

$$1 \geqslant \mu_{nT} 2 + (1-\mu_{nT})0 \Leftrightarrow \mu_{nT} \leqslant \frac{1}{2}.$$

Assuming $\mu_{nT} \leqslant \frac{1}{2}$, if the male grows a tail, type H gets $z_H v + r$ and type L gets $z_L v + r$ whereas if they do not they both get v. As $z_H v + r > z_L v + r$ we only have to check that the weak male does not want to deviate, which occurs whenever

$$z_L v + r \geqslant v \Leftrightarrow r \geqslant (1 - z_L)v,$$

that is, the benefit from mating r must be large enough compared to the cost of growing a tail for a low type.[1]

To summarize if we have

- $p \geqslant \frac{1}{2}$
- $\mu_{nT} \leqslant \frac{1}{2}$
- $r \geqslant (1 - z_L)v$

then the pooling equilibrium ($\boldsymbol{P}_{TT}$), in which both type grow a tale and the female mates when she observes a tail and does not otherwise, exists.

■ **($\boldsymbol{P}_{nn}$) No type grows a tail.** The female's posterior beliefs must then be $\mu_T \in [0,1]$ and $\mu_{nT} = p$.

If the female mates with a tailless male she gets $p2 + (1-p)0 = 2p$ and if she does not she gets 1 for sure. Then she chooses to mate if a tailless male if $p \geqslant \frac{1}{2}$ as in the previous case.

[1]We can indeed say that $(1 - z_K)v$ is the cost of growing a tail for type $K = H, L$. To see why, notice that v is the base payoff without growing a tail and let c_K denote the cost of growing a tail for type $K = H, L$. Then we must have that growing a tail yields $v - c_K = z_K v$ so that $c_K = (1 - z_K)v$ is indeed the cost of growing a tail for type $K = H, L$.

It is then clear that if $p \geqslant \frac{1}{2}$, not growing a tail is a dominant strategy for both types of male. Whatever is $\mu_T \in [0,1]$, growing a tail is only costly for the male.[2]

$(\boldsymbol{P}_{TT} - \mathbf{1})$: A first pooling equilibrium exists when $p \geqslant \frac{1}{2}$. It is such that no type grows a tail and the female mates after observing no tail.

Now assume instead that $p < \frac{1}{2}$. In that case, the female does not want to mate a tailless male. We have therefore to check whether none of the type is willing to deviate by growing a tail. When she observes a male with a tail, the female has belief $\mu_T \in [0,1]$ so that she is willing to mate whenever

$$\mu_T 2 + (1-\mu_T)0 \geqslant 1 \Leftrightarrow \mu_T \geqslant \frac{1}{2}.$$

Assume first that $\mu_T < \frac{1}{2}$, then the females does not want to mate after observing a tail. As $p < \frac{1}{2}$ she does not mate with tailless male either. None of the male has incentive to deviate and we have the following pooling equilibrium.

$(\boldsymbol{P}_{TT} - \mathbf{2})$: If $p < \frac{1}{2}$ and $\mu_T < \frac{1}{2}$, then none of the type grows a tail and the female never mates.

Finally consider the case $p < \frac{1}{2}$ and $\mu_T \geqslant \frac{1}{2}$. The female does not mate with a tailless male but wants to mate with tailed male. Does one of the type wants to deviate and grow a tail? If type $K = H, L$ deviates and grows a tail he gets $z_K v + r$ instead of v. Therefore, no deviation is profitable when

$$z_K v + r < v \Leftarrow r < v(1 - z_K) \text{ for } K = H, L.$$

It is sufficient that this condition is satisfied for the high type, that is $r < v(1 - z_H)$.

$(\boldsymbol{P}_{TT} - \mathbf{3})$: If $p < \frac{1}{2}$, $\mu_T \geqslant \frac{1}{2}$ and $r < v(1 - z_H)$, then no type grows a tail, the fever only mates after observing a tail.

It is interesting to notice that $(\boldsymbol{P}_{TT} - \mathbf{2})$ requires that $\mu_T < \frac{1}{2}$, that is, the female believes that if a male deviates and grows a tail it is more likely that it is a low type. It is a valid PBE, yet one can argue that the required belief to sustain it is not really intuitive.

3. Let us now consider separating equilibria, that is, equilibria in which each type chooses a different action so that their action **reveals** information about their type.

[2] Consider the most favorable case for the male, $\mu_T = 1$, i.e. a female believes that a male with a tail is a high type for sure. Then, she will always choose to mate so that the male always get $z_K v + r$ when of type $K = H, L$. But it would still be profitable not to grow a tail because the female will also mate for sure and type $K = H, L$ would get $v + r > z_K v + r$.

■ **(S_{nTT}) Strong male does not grow a tail, weak male does.** Intuition tells us that this could not be an equilibrium, let us check.

The female's posterior beliefs must be $\mu_T = 0$ and $\mu_{nT} = 1$ in that case. Then, the female will mate with a tailless male and not mate with a tailed male.

There is an obvious deviation here. The weak male will surely not grow a tail to both avoid the cost of growing it and to obtain the benefit of mating. This **cannot be an equilibrium**.

■ **(S_{TnT}) Strong male grows tail, weak male does not.** In that case, the female's posterior beliefs must be $\mu_T = 1$ and $\mu_{nT} = 0$. It is sequentially rational for the female to mate when she observes a tail and not to mate when she does not observe a tail.

We have to check whether none of the type would like to deviate. If the strong male grows a tail (the female will mate), he gets $z_H v + r$ and if he deviates (the female will not mate) he gets v . If the weak make does grow a tail (the female will not mate), he gets v and if he deviates (the female will mate) he gets $z_L v + r$. Therefore, no type is willing to deviate when the two following conditions are satisfied simultaneously satisfied:

$$\begin{aligned} \text{Type: } H \quad & z_H v + r \geqslant v \\ \text{Type: } L \quad & v \leqslant z_L v + r, \end{aligned}$$

Rearranging we get

$$(1 - z_H)v \leqslant r \leqslant (1 - z_L)v.$$

(S_{TnT}) is a separating equilibrium whenever $(1 - z_H)v \leqslant r \leqslant (1 - z_L)v$.

This last condition, $(1 - z_H)v \leqslant r \leqslant (1 - z_L)v$, is the very essence of signaling games. It means that a separating equilibrium exists if, compared to the benefits r, the action that serves as a signal is costly enough for the low type ($r \leqslant (1 - z_L)v$) and not too costly for the high type $(1 - z_H)v \leqslant r$.

In other words, the signaling device (the tail), must be such that low types are not incentivized to use it but that high types are not too severely penalized by it so that they are willing to use it.

4. The idea of *rational handicap* is illustrated here by the separating equilibrium (S_{TnT}) in which the strong male chooses to grow a tail although it is dangerous for him to do so as predators will be more likely to notice him. This is, however, a rational thing to do as the benefits (mating) exceed the cost. The male uses the tail as a signal of strength and this is credible only because it would be too costly for a weak male to grow a tail.

   Obviously, peacocks do not really choose to grow a tail or not in real life but this model captures the idea of an evolutionary process in which for a very large number of generations, this type of natural signaling helped females to mate more often with strong males so that weak males gradually disappear.