Basic of games
Nash Equilibrium

Preliminaries.

Rational individual, acting to maximize

•Objective function (payoff)

by

•Choosing Variable (action/strategy)

Strategic interaction ➔ the payoff of one agent depends on its own actions (strategies) as well as on the actions (strategies) of other agents

Strategic behaviour ➔ the optimal choice (strategy) of each agent depends of what he conjectures other agents will choose

Strategic interaction and ***strategic behaviour*** are widely diffused in any field of the social life: economics, business, politics, diplomacy……….

………**and in our day life**:

Suppose you are invited to a party

You know that many other (nice) people have been invited, but you don’t know how many of them will eventually go

If you go, you enjoy the party only if many of them come (otherwise the party will turn out quite boring).

Your payoff (i.e. how much you will enjoy the party) not only depends on your decision (i.e. whether to accept the invitation), but also on the decisions of the other people invited

In taking your decision, it is very likely that you will try to guess how many people will accept the invitation

Furthermore

other people invited will act like you (i.e. they will try to guess if you will go to the party)

when conjecturing what other people will do, you may need to conjecture what other people conjecture regarding your choice, and so forth.

Also, if the strategic interaction evolves over time (e.g. a sequence of parties with the same people invited), you should also take into account that your decision today can have an impact on the other people conjectures and decisions in the future.

Payoff interdependence (strategic interaction) ➔ a host of possibilities for **strategic behaviour**

Game theory ➔ strategic interaction and behaviour as a game among players

Basic elements of a Game

A game consists of:

- a set of **players**: 1,2,3,…., N
- a set of **strategies** (actions) **for each player**
- a set of **rules**: who can do what and when; players' information
- a set of **payoff functions** (one for each player):

utility (payoff) each player gets as a result of each possible combination of strategies (strategy profile)

➔ a strategy profile is a specific combination of N strategies (one strategy for each player)

➔ a player's payoff function shows the payoff the player would get from each possible strategy profile

- Players are **rational** ➔ aim at maximising their payoff
  pursue this aim in a consistent way
  rational conjectures (beliefs) on other players

Generic definition of a game solution

A solution of a game is a combination of strategies (one for each player, i.e. a strategy profile) such that each player chooses to play his specified strategy according to a sensible notion of rational behaviour.

We call a solution **equilibrium of the game**.

As we will see, we need to specify **different notions** of equilibrium which better apply to different classes of games

For all of them we will stress the corresponding notion of rational behaviour

Game representation – normal form: payoff matrix

Player 2

Player 1		*L*	*R*
	T	5, 5	3, 6
	B	6, 3	4, 4

(In each cell: Player 1's payoff, Player 2's payoff)

Two players: 1, 2

Two strategies for each player

Player 1 strategies:
Top, Bottom

Player 2 strategies:
Left, Right

Four possible strategy profiles: (*T*, *L*) (*T*, *R*) (*B*, *L*) (*B*, *R*)

Each cell gives the payoffs associated to the corresponding profile

(*T*, *L*) ➔ Player 1 gets 5, Player 2 gets 5

(*T*, *R*) ➔ Player 1 gets 3, Player 2 gets 6

….and so on

Simultaneous moves game

The **normal form** representation is the most appropriate when players move simultaneously

Each player must decide his strategy before knowing the other players' decisions

Each player knows all the information summarised in the payoff matrix ➔ other players' strategy sets, all possible strategy profiles and the associated payoffs

First solution concept: equilibrium in dominant strategies

Applicable only to a limited class of games where each player has a dominant strategy

Dominant strategy ➔ **a strategy** giving the player a **higher payoff** than **any other strategy** whatever the rivals will choose

Player 2

Player 1		*L*	*R*
	T	5, 5	3, 6
	B	6, 3	4, 4

Point of view of Player 1

B is better than *T* *either*

if Player 2 plays *L* *or*

if Player 2 plays *R*

B *dominant strategy* for Player 1

Point of view of Player 2

R is better than *L* *either*

if Player 1 plays *T* *or*

if Player 1 plays *B*

R *dominant strategy for* Player 2

In this game each player has a dominant strategy

Rational behaviour ➔ play the dominant strategy

robust notion of individual rational behaviour

each player does not need to assume that the rival is rational (and even to know the rival's payoffs)

Equilibrium in dominant strategies ➔ (*B*, *R*)

		Player 2	
		L	***R***
Player 1	*T*	5, 5	3, 6
	B	6, 3	**4, 4**

Famous game known as ***Prisoner's dilemma***

Playing their respective dominant strategies, (***B***, ***R***) players end up with (***4***, ***4***)

But **both players** would be better off with (***T***, ***L***) giving payoffs (***5***, ***5***)

Why don't they go for (***T***, ***L***) then?

Recall players **cannot decide jointly** what to do (i.e. **cannot cooperate** in selecting a preferred strategy profile). They behave individually.

For each player, the individual incentive is to play his dominant strategy

If player 1 played *T*, player 2 would respond by playing *R*, *not* *L*

Conflict between individual incentives and joint incentives

Nash Equilibrium (general solution concept)

What can we do if there are no dominant or dominated strategies?

In such cases, it is more apparent that each player's optimal strategy depends on his conjectures on which strategies the rivals will choose

Accordingly, the solution concept focuses on players' conjectures

Nash equilibrium (NE) → most powerful solution concept → provides a solution for almost all games (all games we will see in this course)

Intuitive definition of NE → a situation where:

• players choose an optimal strategy given their conjectures on what the other players will do

- **Equivalent definition (more formal and operative)**

NE is a strategy profile (i.e. one strategy for each player) such that the **specified (i.e. equilibrium)** strategy of **each player** is the **best reply** to the **specified (i.e. equilibrium)** strategies of the other players.

Notice ➔ in **equilibrium** each player gets the highest possible payoff conditional on the other players' **equilibrium** strategies

- **Third equivalent definition (no incentive to deviate)**

NE is a strategy profile (i.e. one strategy for each player) such that **no player can unilaterally** change his strategy in a way that **improves** his payoff.

Best Reply Functions (*BRF*)

The second definition, based on the idea of **best reply**, allows to find the NE of a game in a simple way

Consider a game with two players: *A* and *B*

Two strategies for each player: Player *A* ➔ (a_1, a_2) Player B ➔ (b_1, b_2)

Suppose that the strategy profile **(a_1, b_2)** is a Nash Equilibrium

This means that: $\boldsymbol{a_1}$ is the **best strategy (reply) of *A*** when *B* plays $\boldsymbol{b_2}$

$\boldsymbol{b_2}$ is the **best strategy (reply) of *B*** when *A* plays $\boldsymbol{a_1}$

Then, to find a NE:

- *find the best reply of player A to each strategy of player B* (*BRF of A*)
- *find the best reply of player B to each strategy of player A* (*BRF of B*)
- *look for a strategy profile that appears simultaneously in the BRF of both players ("intersection of the Best reply functions of the two players")*

Nash Equilibrium – example

P1 \ P2	*L*	*C*	*R*
T	2, 1	2, 2	0, 1
M	1, 1	1, 2	1, 1
B	0, 2	0, 0	2, 1

***BRF* of P1**

If P2 plays *L* ➔ *T*

If P2 plays *C* ➔ *T*

If P2 plays *R* ➔ *B*

***BRF* of P2**

If P1 plays *T* ➔ *C*

If P1 plays *M* ➔ *C*

If P1 plays *B* ➔ *L*

Only the strategy profile (*T*, *C*) appears in the reply function of both players

(*T*, *C*) is the **only cell** where we find both a red **and** a blue circle

(*T*, *C*) is the unique Nash Equilibrium of this game.

Multiple Nash Equilibria

NE solves almost all games. But the solution is not always unique. One example of multiple Nash equilibria is the following *coordination game*

		Pl 2	
		Yes	*No*
Pl 1	*Yes*	1 , 2	0 , 0
	No	0 , 0	2 , 1

Solving the game:
Both (*Yes*, *Yes*) and (*No*, *No*) are Nash equilibria

Players must (separately) answer *Yes* or *No* to the same question

Both are better off if the answer is the same (*coordination*)

But

P1 prefers *coordination* on *No*,
P2 prefers *coordination* on *Yes*

(*disagreement on the point of coordination*)

Sequential moves games

Frequently strategic interaction involves some players choosing their strategies after having observed the decision of other players.

In this case:

- players move sequentially
- the *strategies* of the *players who move after* are *conditional actions* which depend on the *observed actions* of the players who move *first*

Example (entry game)

Situation: an industry monopolised by an *Incumbent (I)* firm but with a potential *Entrant (E)*

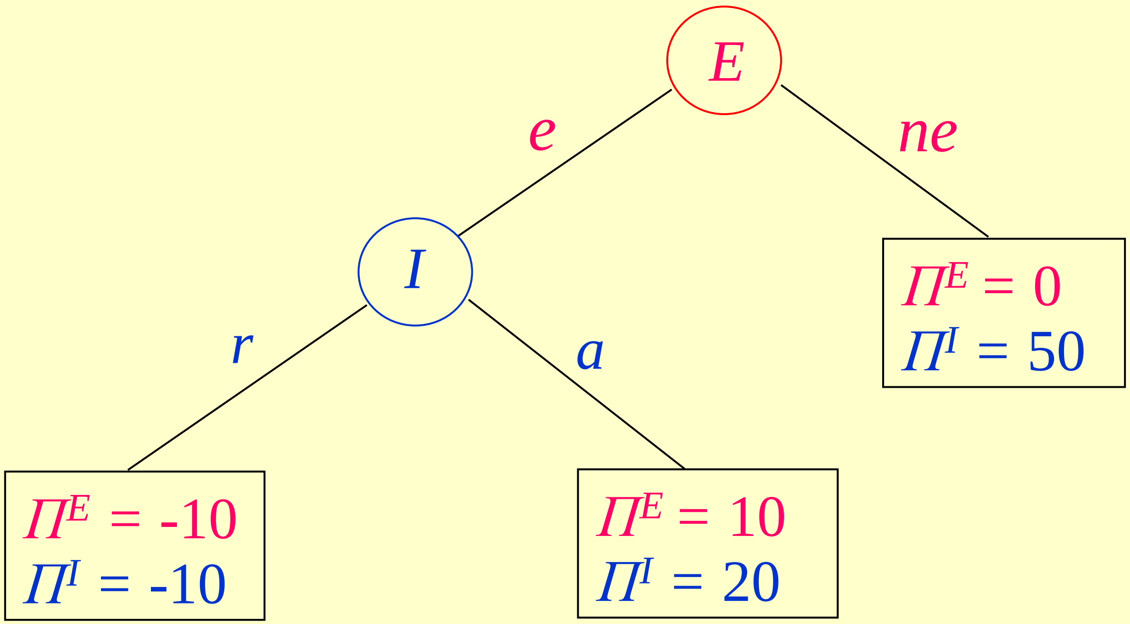
E moves first. Two possible strategies (actions): enter (*e*); not enter (*ne*)

I moves second. If *E* enters, *I* can choose between retaliate (*r*) or accommodate (*a*) the entry.
If *E* does not enter, the game is over.

Game tree (extensive form representation).

Sequential games are usually represented with a game tree.

Entry game tree



Decision nodes (circles)

Timing

Branches ➔ strategies

End nodes (squares) ➔ payoffs

Entry game – tentative solution applying NE without other qualifications. With sequential games, this leads to a problem

Let us represent the entry game in normal form (payoff matrix)

		Incumbent	
		$r(e)$	$a(e)$
Entrant	e	-10, -10	10, 20
	ne	0, 50	0, 50

BRF of E

If I plays $r(e)$ ➔ ne

If I plays $a(e)$ ➔ e

BRF of I

If E plays e ➔ $a(e)$

If E plays ne ➔ $a(e)$ *or* $r(e)$

Two Nash Equilibria: $(e, a(e))$ $(ne, r(e))$

$(10, 20)$ $(0, 50)$

But the equilibrium $(ne, r(e))$ is not reasonable.

Supported by the **not credible threat** of I to **r**etaliate if E **e**nters the market

With sequential games, applying *NE* without additional requirements can lead to not reasonable outcomes.

Subgame Perfect Equilibrium (*SPE*): Nash equilibrium which is not supported by not credible threats (i.e. sub-optimal actions threatened by some player out of the equilibrium path)

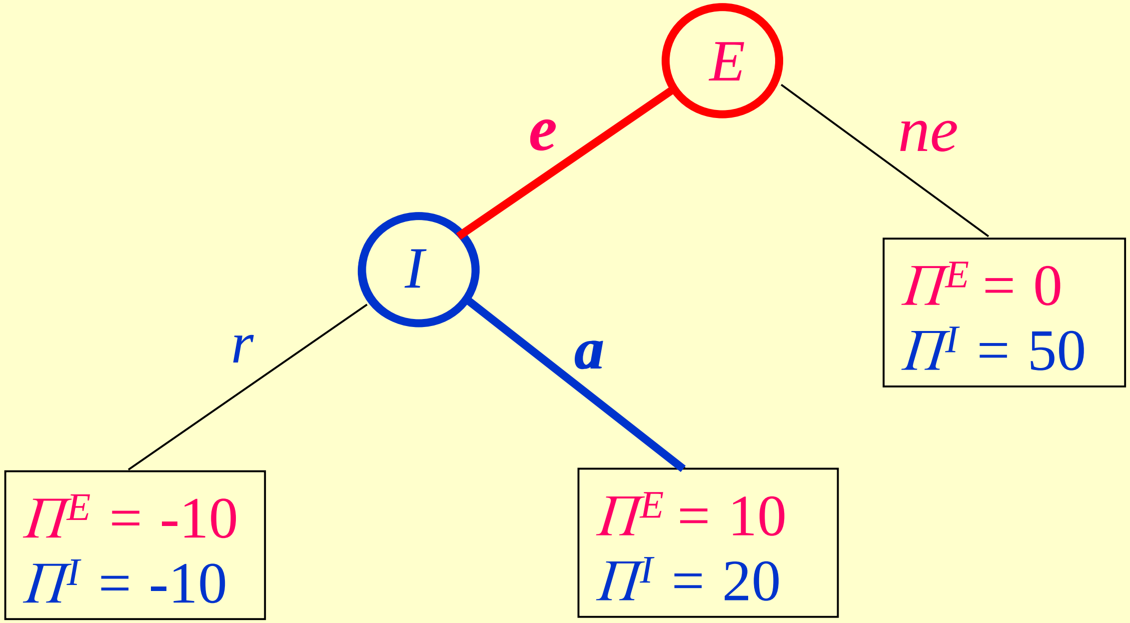
Why "subgame" perfect equilibrium ?

To get rid of not credible threats, we require that each player chooses optimal strategies from any decision node of the game tree.
That is, optimal strategies in the *continuation* of the game following any decision node (*subgame*).

Backward induction ➔ start from the last decision node(s). Find the optimal choice of the player(s) moving last (given the previous history of the game leading to that last node).

Then, go back to the previous node(s), and find the optimal choice of the player(s) moving before, **given** the optimal choice of the player(s) moving last. And so on, up to the first node.

Entry game – backward induction ➔ *SPE*



Start from the last node

If *E* enters, *I* prefers ***a***
payoff **20** instead of -10

Back to the first node

Since *I* will optimally reply ***a*** to the entry,
E prefers ***e***
payoff **10** instead of 0

Backward induction

➔ eliminates not credible threats

➔ selects the reasonable *NE* ➔ *SPE* ➔ (***e***, ***a*(*e*)**)

Summing-up

When players move sequentially:

- a reasonable equilibrium cannot contain not-credible threats ➔ ***SPE***
- to find a *SPE* we must solve the game backward ➔ **backward induction**

Sequential games can be more complicated.

At some (or all) nodes, instead of having only one player taking a decision we can have a *proper subgame* (like a simultaneous moves game) played by several players

Example: two firms

first (simultaneously) decide their productive capacity (first stage of the game)

after capacities are set (and observed by both firms), they simultaneously choose their price (second stage of the game)

What does determine which kind of decision come first?

Backward induction must be used also for these games ➔ *SPE*

But the procedure is a little bit more complicated.

In our example (capacities first, prices second):

Start from the final stage of the game (price subgame). Find the ***NE* prices of the two firms** given (generic) values for the productive capacities

The **equilibrum** prices and payoffs of the price subgame will be functions of the productive capacities ➔ payoff functions for the first stage subgame

Solve the capacity subgame (first stage) using the payoff functions found before ➔ **NE productive capacities of the two firms.**

Use the equilibrium capacities to specify the equilibrium values of prices