Exam Statistics 2^{nd} February 2023 (A)

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This is a closed book exam. Answer all the following questions and solve all the following exercises. You have two hours to complete the exam.

Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables. Let $f_\theta(x)$ be the density function. Let $\hat{\theta}$ be the MLE of θ, θ_0 be the true parameter, $L(\theta)$ be the likelihood function, $l(\theta)$ be the loglikelihood function, and $I(\theta)$ be the Fisher information matrix

1. Which of the following is not true?

   a
   $$u(\hat{\theta}) = 0$$

   b
   $$E\left[\frac{\partial l(\theta|x)}{\partial \theta}\right] = 0$$

   c
   $$Var\left(\frac{\partial logL(\theta|x)}{\partial \theta}\right) = -E\left(\frac{\partial^2 logL(\theta|x)}{\partial^2 \theta}\right)$$

   d
   $$E\left[\left(\frac{\partial logL(\theta|x)}{\partial \theta}\right)^2\right] = E\left(\frac{\partial^2 logL(\theta|x)}{\partial^2 \theta}\right)$$

2. In point estimation:

   a the mean of the population equals the mean of the sample

   b data from the population is used to estimate the population parameter

   c data from the sample is used to estimate the population parameter

   d data from the sample is used to estimate the sample statistic

3. Failing to reject the null hypothesis when it is false is:

   a Type I error

   b Type II error

   c β

   d α

4. An estimator $T = T(X)$ is a consistent estimator for θ if

   a The distribution of T does not depend on θ.

   b The more data you collect, the closer the estimate will be to the real population parameter.

   c The distribution of T is asymptotically Gaussian

   d It is an accurate statistic that's used to approximate a population parameter.

Exercise 2

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \theta(1-x)^{\theta-1} I_{(0,1)}(x) \qquad \theta > 1$$

1. Find a sufficient statistics for θ

2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.

3. Compute the score function and the Fisher information.

4. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.

5. Suppose form a random sample of 250 random variables you obtain $\sum_{i=1}^{250} log\,(1-x_i) = -100$. Let $\alpha = 0.01$, Complete ALL of the following questions:

   (a) Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the asymptotic distribution and verify the null hypothesis.

   (b) Find the Wald test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.

   (c) Find the Score test statistic for testing $H_0 : \theta = 2$ versus $H_1 : \theta \neq 2$, specify the distribution and verify the null hypothesis.

Exercise 3

Let $(X_1,\, X_2,\, X_3,\, X_4)$ be four random variables distributed as a $Uniform(0, \theta)$ with parameter θ unknown, suppose we are interested to test $H_0 : \theta = 4$ versus $H_1 : \theta = 2$ and consider the following rejection region

$$R = \{(x_1,\, x_2,\, x_3,\, x_4) : max_i(x_i) \leq 3\}$$

Find α and β respectively probability of type I and type II errors.

Exercise 4

Let $(X_1, \ldots, X_n)$ be independent identically distributed random variables with $E(X) = \mu$, $Var(X) = \sigma^2$, Is the following estimator unbiased estimator for σ^2?

$$T(X_1, \ldots, X_n) \quad = \quad \frac{(\sum_{j=1}^n X_j)^2}{n} - X_1(\sum_{j=2}^n X_j) - X_2X_3$$

Exercise 5

Suppose that $X_1, X_2, \ldots, X_n$ form a random sample from a Poisson distribution with parameter λ consider the following statistics

$$T = \frac{\sum_i I(X_i = 0)}{n}$$

1. Is T an unbiased estimator of $\theta = exp(-\lambda)$?
2. Can you provide an other unbiased estimator for $\theta = exp(-\lambda)$ with a lower variance?
3. Suppose you have a sample of size 4 observe the following values $(x_1 = 4,\, x_2 = 0,\, x_3 = 0,\, x_4 = 5)$ provide an unbiased estimate for $exp(-\lambda)$

Exercise 6

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables with expected value μ and variance σ^2. Consider the following estimator of μ:

$$T_n(a) = a \times X_n + (1 - a) \times \bar{X}_{n-1}$$

where X_n is the $n - th$ observed random variable and $\bar{X}_{n-1}$ is the sample mean based on $n - 1$ observations.

1. Find value of a such that $T_n(a)$ is an unbiased estimator for μ

2. Find value of $a\star$ such that $T_n(a\star)$ is the most efficient estimator for μ within the class $T_n(a)$?

3. Define concept of efficiency

Exercise 7

Provide correct statement for Neyman Pearson Lemma