Gaussian distribution

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Sufficient Statistics

$$f(y_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2}$$

σ^2 known, Sufficient Statistics for μ

$$f(y_1,\ldots,y_i,\ldots,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i-\mu\right)^2}$$

$$\sum_{i=1}^{n}\left(y_i-\mu\right)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2}\, e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

σ^2 known, Sufficient Statistics for μ

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

$$T(y) = \sum_{i=1}^{n} y_i \ \ \textit{sufficient statistics for } \mu$$

Sufficient Statistics for σ^2

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

Sufficient Statistics for σ^2

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

Sufficient Statistics for σ^2

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2}\, e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i}\, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu}$$

$$T(y) = \left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} y_i^2\right) \quad \textit{sufficient statistics for } \sigma^2$$

Sufficient Statistics for μ and σ²

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

Sufficient Statistics for μ and σ²

$$f(y_1,..,y_i,..,y_n \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$f(y \mid \mu,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2} e^{+\frac{1}{\sigma^2}\mu\sum_{i=1}^{n} y_i} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\mu^2}$$

$$T(y) = \left(\sum_{i=1}^{n} y_i \,, \sum_{i=1}^{n} y_i^2\right) \ \textit{sufficient statistics for } (\mu,\sigma^2)$$

Maximum Likelihood estimation σ^2 known

$$f(y_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2}$$

Maximum Likelihood estimation σ^2 known

$$L(\mu,\sigma^2|y_i) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$l(\mu,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{dl(\mu,\sigma^2)}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu)$$

Maximum Likelihood estimation for μ and (σ^2 known)

$$0 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \mu)$$

$$\hat{\mu} = \bar{y}$$

$$\frac{d^2 l(\mu,\sigma^2)}{d\mu^2} = -\frac{n}{\sigma^2} < 0$$

Maximum Likelihood estimation

$$f(y_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}\left(y_i - \mu\right)^2}$$

$$f(y_1, .., y_i, .., y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)^2}$$

Maximum Likelihood estimation

$$L(\mu,\sigma^2 \mid y_i) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$$

$$l(\mu,\sigma^2) = -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

$$\frac{\partial l(\mu,\sigma^2)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu)$$

$$\frac{\partial l(\mu,\sigma^2)}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(y_i-\mu)^2$$

$u(\mu,\sigma^2)$ score function

$$\frac{\partial l(\mu,\sigma^2)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)$$

$$\frac{\partial l(\mu,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\left(y_i - \mu\right)^2$$

Second Derivative

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial (\sigma^2)^2} = +\frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}\left(y_i - \mu\right)^2$$

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu \, \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(y_i - \mu\right)$$

Second Derivative

$$E\left(\frac{\partial^2 l(\mu,\sigma^2)}{\partial\mu^2}\right) = -\frac{n}{\sigma^2}$$

$$E\left(\frac{\partial^2 l(\mu,\sigma^2)}{\partial(\sigma^2)^2}\right) = -\frac{n}{2\sigma^4}$$

$$E\left(\frac{\partial^2 l(\mu,\sigma^2)}{\partial\mu\,\partial\sigma^2}\right) = 0$$

Second Derivative

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial (\sigma^2)^2} = +\frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(y_i - \mu)^2 = -\frac{n}{2\sigma^4}$$

$$\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu \, \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}(y_i - \mu)$$

Maximum Likelihood estimation

$$0 = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - \mu\right)$$

$$0 = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\left(y_i - \mu\right)^2$$

$$\hat{\mu} = \bar{y}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}{n}$$

Maximum Likelihood estimation

$$\hat{\mu} = \bar{y}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2}{n}$$

Check Second Derivatives?

Second Derivative

$$\left.\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu^2}\right|_{\hat{\mu},\hat{\sigma}^2} = -\frac{n}{\hat{\sigma}^2}$$

$$\left.\frac{\partial^2 l(\mu,\sigma^2)}{\partial (\sigma^2)^2}\right|_{\hat{\mu},\hat{\sigma}^2} = +\frac{n}{2\hat{\sigma}^4} - \frac{1}{\hat{\sigma}^6}\sum_{i=1}^{n}\left(y_i - \hat{\mu}\right)^2 = -\frac{n}{2\hat{\sigma}^4}$$

$$\left.\frac{\partial^2 l(\mu,\sigma^2)}{\partial \mu \, \partial \sigma^2}\right|_{\hat{\mu},\hat{\sigma}^2} = -\frac{1}{\hat{\sigma}^4}\sum_{i=1}^{n}\left(y_i - \hat{\mu}\right) = 0$$

The Determinant of Hessian matrix of the second-order partial derivative

$$\begin{vmatrix} -\frac{n}{\sigma^2} & 0 \\ 0 & -\frac{n}{2\sigma^4} \end{vmatrix} = \frac{n^2}{2\sigma^6}$$

The Determinant of Hessian matrix of the second-order partial derivative is positive!

$I(\mu,\sigma^2)^{-1}$
Inverse of Fisher Information

$$\begin{bmatrix} \dfrac{\sigma^2}{n} & 0 \\ 0 & \dfrac{2\sigma^4}{n} \end{bmatrix}$$

Asymptotic distribution mle

$\left(\mu_0, \sigma_0{}^2\right)$ real value of parameters

$$\sqrt{n}\left(\left(\hat{\mu}, \hat{\sigma}^2\right) - \left(\mu_0, \sigma_0{}^2\right)\right) \to_{n\to\infty} N\left(\left(0,0\right), \begin{bmatrix} \sigma_0{}^2 & 0 \\ 0 & 2\sigma_0{}^4 \end{bmatrix}\right)$$

$$\left(\hat{\mu}, \hat{\sigma}^2\right) \approx N\left(\left(\mu_0, \sigma_0{}^2\right), \begin{bmatrix} \dfrac{\sigma_0{}^2}{n} & 0 \\ 0 & \dfrac{2\sigma_0{}^4}{n} \end{bmatrix}\right)$$