

# Testing Hypothesis

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# Hypothesis Testing

*"It is a mistake to confound strangeness with mystery"*

**Sherlock Holmes**  
A Study in Scarlet

# Outline

- 1 General Concept
  - Definitions
  - The Power Function
  - Methods of evaluating tests
- 2 Neyman-Pearson Lemma
  - Neyman-Pearson Lemma
  - Likelihood Ratio Test
- 3 Hypothesis Testing:  $N(\mu, \sigma^2)$
- 4 Method of inversion

# General Concept

## Definition

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For example:

- population mean
- population proportion
- population variance
- distribution

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- population proportion:  $H_0 : \pi \geq 0.5$  versus  $H_1 : \pi < 0.5$

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A test of a hypothesis is a process that uses sample statistics to test a claim about the value of a population parameter and to decide, which of two complementary hypotheses, in a hypothesis testing problem, is true



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Two complementary hypothesis in a hypothesis testing problem are called the null and the alternative hypothesis

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# What's a Hypothesis Test?

**Definition** A hypothesis procedure or hypothesis test is a rule that specifies:

- For which sample values the decision is made to accept  $H_0$  as true
- For which sample values  $H_0$  is rejected and  $H_1$  is accepted as true

# Rejection Region

The subset of the sample space for which  $H_0$  will be rejected is called the rejection region or critical region. The complement of the rejection region is called the acceptance region.

The rejection region  $R$  of a hypothesis test is usually defined by a test statistic  $W(X)$ , a function of the sample, for example:

$$R = \{X : W(X) > c\} \Rightarrow \text{reject } H_0$$

$$A = \{X : W(X) \leq c\} \Rightarrow \text{accept } H_0$$

# Statistical rule

## Procedure

Given a random sample  $X_1, X_2, \dots, X_n$  a test statistic  $W$  is a function of the random sample  $W(X) = W(X_1, X_2, \dots, X_n)$  such that

- if  $W \in R$  then  $H_0$  is rejected
- if  $W \in A$  then  $H_0$  is not rejected

$R$  is called critical or rejection region.

# Types of Errors

- Since the decision is based on a sample and not on the entire population, there is always the possibility you'll make the wrong decision.
- There are two types of errors.
  - A type I error occurs if  $H_0$  is rejected when it is actually true
  - A type II error occurs if  $H_0$  is not rejected when it is actually false

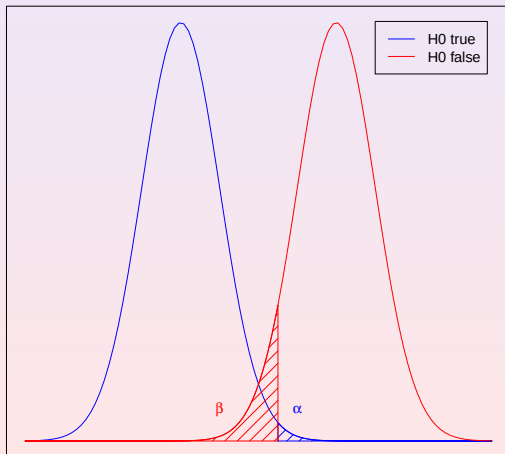
# Types of Errors

## Type I and II Errors

| Decision→<br>Reality↓ | $H_0$ is rejected        | $H_0$ is not rejected    |
|-----------------------|--------------------------|--------------------------|
| $H_0$ true            | type I error<br>$\alpha$ | no error<br>$1 - \alpha$ |
| $H_0$ false           | no error<br>$1 - \beta$  | type II error<br>$\beta$ |

# Type I and II errors

Let  $H_0: \theta = \theta_0$  vs  $H_1: \theta = \theta_1$ . Reject  $H_0$  if  $W > c$



# Errors in making Decisions

## Type I Error

- Reject true null hypothesis
- $\alpha = P(\text{reject } H_0 | H_0 \text{ true})$
- Has serious consequences: type I error is considered more dangerous, we do our best to not commit type I error
- Probability of Type I Error is denoted with  $\alpha$
- Called **Level of Significance**

# Errors in making Decisions

## Type II Error

- Do not reject false null hypothesis
- $\beta = P(\text{not reject } H_0 | H_0 \text{ false})$
- It is considered less serious
- Probability of Type II Error is denoted with  $\beta$



## Remarks

- The significance level of a test is predetermined, typically set  $\alpha = 0.01$  or  $\alpha = 0.05$ , although values such as  $\alpha = 0.10$  are sometimes accepted
- Once  $\alpha$  has been specified, the corresponding critical region  $C$  is determined, and  $C$  depends on the chosen significance level
- Since a test of significance controls only the probability of a Type I error, the probability of committing a Type II error may be relatively large
- For this reason, a test is considered useful only when it leads to the rejection of the null hypothesis ( $H_0$ ). If the test fails to reject  $H_0$ , its result is often regarded as inconclusive, because the Type II error might be substantial.

# Tests and Trials

## Trials

In the courtroom, juries must make a decision about the guilt or innocence of a defendant. Suppose you are on the jury in a murder trial. It is obviously a mistake if the jury claims the suspect is guilty when in fact he or she is innocent. What is the other type of mistake the jury could make? Which is more serious?

## Decision Results

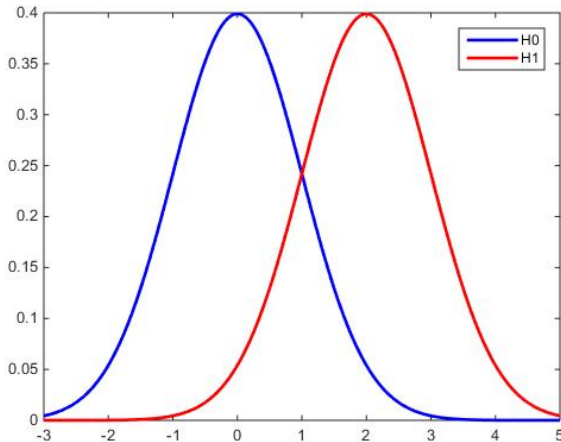
| Jury Trial |                  |         | $H_0$ Test   |                           |                           |
|------------|------------------|---------|--------------|---------------------------|---------------------------|
| Verdict    | Actual Situation |         | Decision     | Actual Situation          |                           |
|            | Innocent         | Guilty  |              | $H_0$ True                | $H_0$ False               |
| Innocent   | Correct          | Error   | Accept $H_0$ | $1 - \alpha$              | Type II Error ( $\beta$ ) |
| Guilty     | Error            | Correct | Reject $H_0$ | Type I Error ( $\alpha$ ) | Power ( $1 - \beta$ )     |

# Example $N(\mu, \sigma = 10)$ , $n=100$

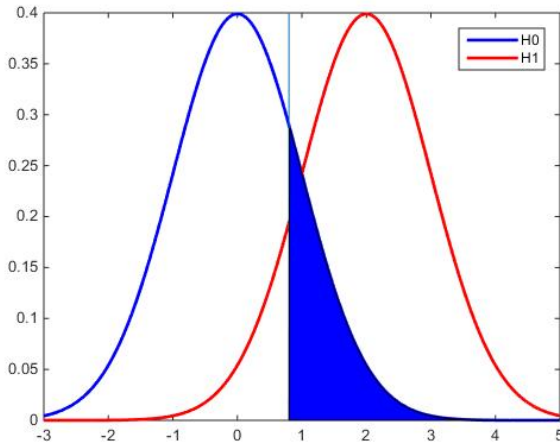
$$H_0 : \mu = 0 \quad \text{vs} \quad H_1 : \mu = 2$$

$$R = \{x : \bar{x} > 0.8\}$$

# Example $N(\mu, \sigma = 10)$ , $n=100$



Example  $N(\mu, \sigma = 10)$ ,  $n=100$ ,  $\alpha$



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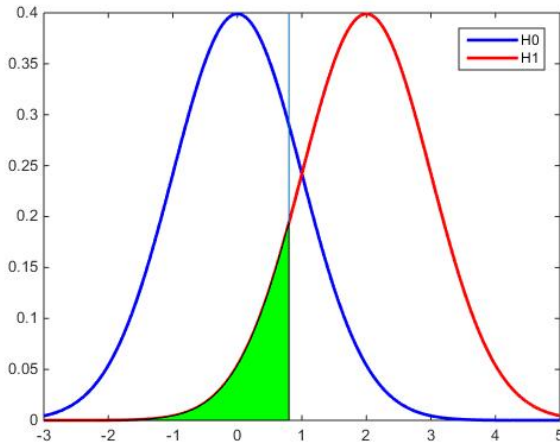
$$\alpha = P(\bar{X} > 0.8 | H_0)$$

$$\alpha = P\left(\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > \frac{0.8 - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$\alpha = P\left(\frac{\bar{X} - 0}{1} > 0.8\right)$$

$$\alpha = 0.2119$$

Example  $N(\mu, \sigma = 10)$ ,  $n=100$ ,  $\beta$





Example  $N(\mu, \sigma = 10)$ ,  $n=100$ ,  $\beta$

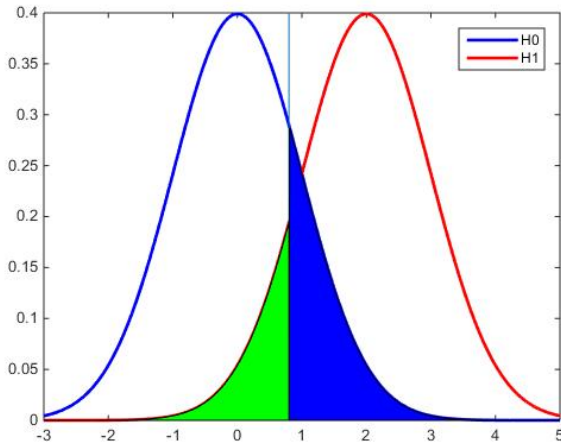
$$\beta = P(\bar{X} \leq 0.8 | H_1)$$

$$\beta = P\left(\frac{\bar{X} - \mu_1}{\frac{\sigma}{\sqrt{n}}} \leq \frac{0.8 - \mu_1}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$\beta = P\left(\frac{\bar{X} - 2}{1} \leq -1.2\right)$$

$$\beta = 0.1151$$

# Example $N(\mu, \sigma = 10)$



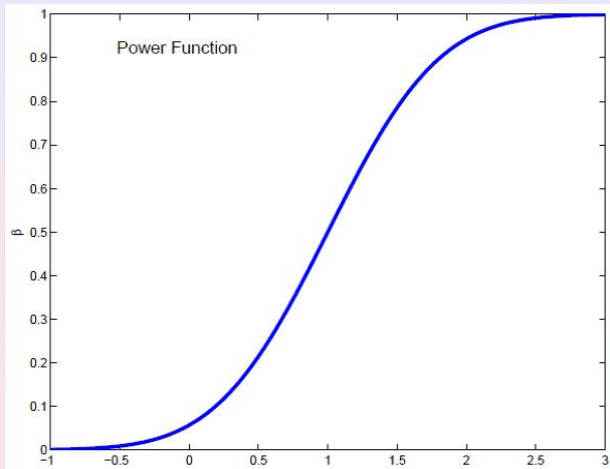
# The Power Function

**Definition:** The *power function* of a hypothesis test with rejection region  $R$  is the function of  $\theta$  defined by:  
$$\beta(\theta) = P_{\theta}(\mathbf{X} \in \mathcal{R})$$

**Example:** Let  $X_1, \dots, X_{10}$  be a random sample from a population  $N(\mu, \sigma^2 = 4)$  and let  $H_0 : \mu \leq 0$  and  $H_1 : \mu > 0$  and let consider a test that reject  $H_0$  when  $\bar{X} > 1$ , the power function has the following expression:

$$\beta(\mu) = P(\bar{X} > 1 | \mu = \mu) = P\left(Z > \frac{1 - \mu}{\frac{2}{\sqrt{10}}}\right)$$

# The Power Function



# $\alpha$ and $\beta$

**Definition** For  $0 \leq \alpha \leq 1$ , a test with power function  $\beta(\theta)$  is a size  $\alpha$  test if  $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$

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## Example: Computation of $\alpha$ and $\beta$

Let  $X_1, \dots, X_3$  be a random sample from a population  $N(\mu, \sigma^2 = 4)$  and let  $H_0 : \mu = 0$  and  $H_1 : \mu = 1$  and let consider a test with the following rejection region:

$$R = \{(x_1, x_2, x_3) : x_1 + x_2 - x_3 > 0.8\}$$

Calculate  $\alpha$  and  $\beta$

# $\alpha$ and $\beta$

$$\alpha = P(X_1 + X_2 - X_3 > 0.8 | \mu = 0) \text{ and}$$

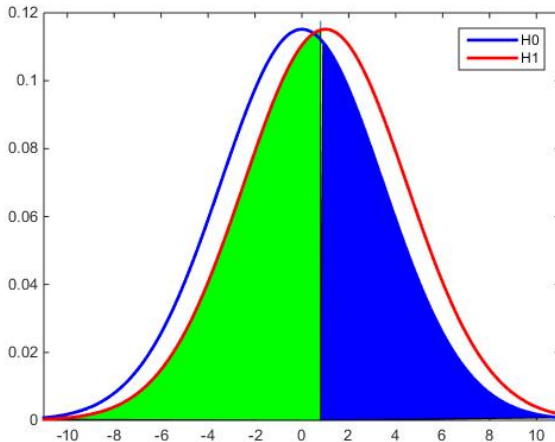
$$\beta = P(X_1 + X_2 - X_3 \leq 0.8 | \mu = 1)$$

$$X_1 + X_2 - X_3 \sim N(\mu, 3 \times \sigma^2)$$

$$\text{Let } Y = X_1 + X_2 - X_3$$

- Under null hypothesis:  $Y \sim N(0, 12)$
- Under alternative hypothesis:  $Y \sim N(1, 12)$

# Example





# $\alpha$ and $\beta$

$$X_1 + X_2 - X_3 \sim N(\mu, 3 \times \sigma^2)$$

$$\text{Let } Y = X_1 + X_2 - X_3$$

- Under null hypothesis:  $Y \sim N(0, 12)$
- Under alternative hypothesis:  $Y \sim N(1, 12)$

$$\alpha = P(Y > 0.8 | \mu = 0) = P\left(\frac{Y-0}{\sqrt{12}} > \frac{0.8-0}{\sqrt{12}}\right) = P(Z > 0.23) = 0.409$$

$$\beta = P(Y \leq 0.8 | \mu = 1) = P\left(\frac{Y-1}{\sqrt{12}} \leq \frac{0.8-1}{\sqrt{12}}\right) = P(Z \leq -0.058) = 0.477$$

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$$R = \left\{ (x_1, x_2, x_3) : \frac{x_1 + x_2 + x_3}{3} > 0.8 \right\}$$

Calculate  $\alpha$  and  $\beta$

# $\alpha$ and $\beta$

$$\alpha = P\left(\frac{X_1+X_2+X_3}{3} > 0.8 \mid \mu = 0\right) \text{ and}$$

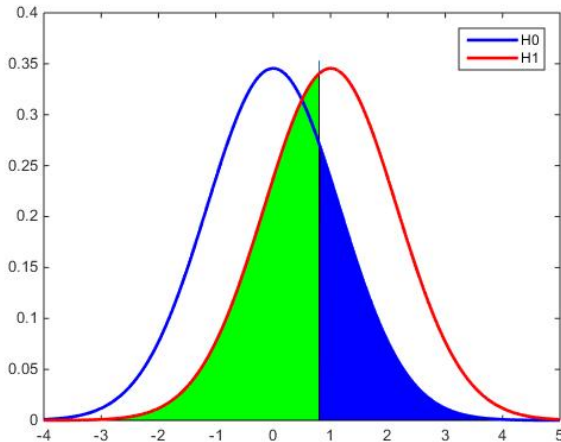
$$\beta = P\left(\frac{X_1+X_2+X_3}{3} \leq 0.8 \mid \mu = 1\right)$$

$$\frac{X_1+X_2+X_3}{3} \sim N\left(\mu, \frac{\sigma^2}{3}\right)$$

$$\text{Let } \frac{X_1+X_2+X_3}{3}$$

- Under null hypothesis:  $Y \sim N\left(0, \frac{4}{3}\right)$
- Under alternative hypothesis:  $Y \sim N\left(1, \frac{4}{3}\right)$

# Example



$\alpha$  and  $\beta$ 

$$\alpha = P(Y > 0.8 | \mu = 0) = P\left(\frac{Y-0}{\sqrt{4/3}} > \frac{0.8-0}{\sqrt{4/3}}\right) = 0.244$$

$$\beta = P(Y \leq 0.8 | \mu = 1) = P\left(\frac{Y-1}{\sqrt{4/3}} \leq \frac{0.8-1}{\sqrt{4/3}}\right) = 0.431$$

## Most Powerful Tests

**Definition:** The region  $C$  is a uniformly most powerful critical region of size  $\alpha$  for testing the simple hypothesis  $H_0$  against a composite alternative hypothesis  $H_1$  if  $C$  is a best critical region of size  $\alpha$  for testing  $H_0$  against each simple hypothesis in  $H_1$ . The resulting test is said to be uniformly most powerful.

## Most Powerful Tests

**Definition:** Let  $C$  be a class of tests for testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta = \theta_1$  of size  $\alpha$ . A test in class  $C$ , with power function  $\beta(\theta)$ , is a uniformly most powerful (UMP) class  $C$  test if  $\beta(\theta) \geq \beta'(\theta)$  for every  $\theta \in \Theta_0^c$  and for every  $\beta'(\theta)$  that is a power function of a test in class  $C$ .

# Evaluating a Test

## Size of a Test

For  $0 \leq \alpha \leq 1$ , a test with power function  $\beta(\theta)$  is a size- $\alpha$  test if  $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$

## Uniformly Most Powerful Test of size $\alpha$

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## Neyman-Pearson Lemma

**Theorem:** Consider testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta = \theta_1$  where the pdf or pmf corresponding to  $\theta_i$  is  $f(x|\theta_i)$ ,  $i = 0, 1$  using a test with rejection region  $R$  that satisfies

$$x \in R \quad \text{if } f(x|\theta_1) > kf(x|\theta_0)$$

and

$$x \in R^C \quad \text{if } f(x|\theta_1) \leq kf(x|\theta_0)$$

and

$$\alpha = P_{\theta_0}(X \in R)$$

Any test that satisfies previous conditions is a **UMP level  $\alpha$  test**.

## Corollary of Neyman-Pearson

**Corollary:** Consider testing  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta = \theta_1$ . Suppose  $T(\mathbf{X})$  is a sufficient statistics for  $\theta$  and  $g(x|\theta_i)$ ,  $i = 0, 1$  is the pdf or pmf corresponding to  $\theta_i$ . Then any test statistics based on  $T$  with rejection region  $S$  is a UMP level  $\alpha$  test if it satisfies

$$t \in S \quad \text{if } g(t|\theta_1) > kg(t|\theta_0)$$

and

$$t \in S^C \quad \text{if } g(t|\theta_1) \leq kg(t|\theta_0)$$

and

$$\alpha = P_{\theta_0}(T \in S)$$

## Application Neyman-Pearson Lemma

- Suppose  $X$  has an exponential distribution with arrival rate  $\lambda$ . We wish to test the hypothesis that  $\lambda = 1$  against the alternative that  $\lambda = 2$ .
- Suppose  $X$  has a Gaussian distribution with mean  $\mu$  and known variance. We wish to test the hypothesis that  $\mu = \mu_0$  against the alternative that  $\mu = \mu_1$ .

## LRT and UMP test

**Theorem (UMP tests)** If the LR test of size  $\alpha$  for  $H_0 : \theta = \theta_0$  versus the simple alternative  $H_0 : \theta = \theta_1$  is the same test for all  $\theta_1 \in \Theta_1$ , (for example  $\Theta_1 = (\theta_1, \infty)$ ) then this is the UMP test of size  $\alpha$  for  $H_0 : \theta = \theta_0$  against the compound alternative  $H_0 : \theta \in \Theta_1$ . Otherwise no UMP test of these hypotheses of size  $\alpha$  exists.

**Remark** So whenever an UMP test exists for a simple  $H_0$  it is a LR test.

# Generalized Likelihood Ratio Tests

We have been looking at really simple problems, typically with just one parameter, but in the real world of applied statistics we constantly deal with models having many parameters. In that context we never have a simple null hypothesis. Our null hypotheses often assert that one of the parameters takes a specific value, but such a hypothesis is not simple, since that would entail specifying the values of all the parameters.

# Likelihood Ratio Test

Definition: The **likelihood ratio test statistics** for testing  $H_0 : \theta \in \Theta_0$  versus  $H_1 : \theta \in \Theta_0^C$  is

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{x})}{\sup_{\Theta} L(\theta|\mathbf{x})}, \quad \Theta = \Theta_0 \cup \Theta_0^C$$

Definition: The **likelihood ratio test** (LRT) is any test that has a rejection region of the form :  $\{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$  where  $c$  is any number satisfying  $0 \leq c \leq 1$ .

This should be reduced to the simplest possible form.

## Likelihood Ratio Tests

The rationale behind LRTs may best be understood in the situation in which  $f(x|\theta)$  is the probability distribution of a discrete random variable. In this case, the numerator of  $\lambda(\mathbf{x})$  is the maximum probability of the observed sample, maximum being computed under the null hypothesis. On the other hand, the denominator of  $\lambda(\mathbf{x})$  is the maximum probability of the observed sample over all possible parameters.



## Likelihood Ratio Tests

The ratio of these two maxima is small if there are parameter points in  $H_1$  for which the observed sample is much more likely than for any parameter in  $H_0$ . In this situation, the LRT criterion says  $H_0$  should be rejected and  $H_1$  accepted as true.

## Relation between LRT and MLE

Suppose MLE of  $\theta$  exists. Let  $\hat{\theta}_0$  be the MLE of  $\theta$  in the null set  $\Theta_0$  (restricted maximization) and  $\hat{\theta}$  be the MLE of  $\theta$  in the full set  $\Theta$  (unrestricted maximization). Then the LRT statistic, a function of  $\mathbf{x}$  (not  $\theta$ ) is

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta|\mathbf{x})}{\sup_{\Theta} L(\theta|\mathbf{x})} = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}$$

In  $R = \{\mathbf{x} : \lambda(\mathbf{x}) \leq c\}$ , different  $c$  gives different rejection region and hence different tests.

## LRT and sufficiency

**Theorem** If  $T(X)$  is a sufficient statistic for  $\theta$ ,  $\lambda^*(t)$  is the LRT statistic based on  $T$ , and  $\lambda(x)$  is the LRT statistic based on  $x$ . Then

$$\lambda^*(T(x)) = \lambda(x)$$

for every  $x$  in the sample space.

Thus the simplified expression for  $\lambda(x)$  should depend on  $x$  only through  $T(x)$  if  $T(X)$  is a sufficient statistic for  $\theta$ .

## Example: Likelihood ratio test

Let  $X_1, X_2, \dots, X_n$  be a random sample from an exponential population with pdf:

$$f(x|\theta) = \begin{cases} \exp(-(x - \theta)) & x \geq \theta \\ 0 & x < \theta \end{cases}$$

The likelihood function is:

$$L(\theta|x) = \begin{cases} \exp(-\sum_i x_i + n\theta) & \theta \leq x_{(1)} \\ 0 & \theta > x_{(1)} \end{cases}$$

## Example: Likelihood ratio test

Consider testing  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$  where  $\theta_0$  is a value specified. The likelihood function is an increasing function of  $\theta$  on  $-\infty < \theta \leq x_{(1)}$ . The likelihood ratio test statistics is

$$\lambda(\mathbf{x}) = \begin{cases} 1 & x_{(1)} \leq \theta_0 \\ \exp(-n(x_{(1)} - \theta_0)) & x_{(1)} > \theta_0 \end{cases}$$

An LRT, a test that rejects  $H_0$  if  $\lambda(\mathbf{X}) \leq c$ , is a test with rejection region  $\{\mathbf{x} : x_{(1)} \geq \theta_0 - \frac{-\log c}{n}\}$ . Note that the rejection region depends on the sample only through the sufficient statistics  $X_{(1)}$ .

# Likelihood Ratio Tests and Newman Pearson lemma

The Neyman-Pearson Lemma asserts that, in general a best critical region can be found by finding the  $n$ -dimensional points in the sample space for which the likelihood ratio is smaller than some constant.

## Monotone Likelihood Ratio (MLR)

**Definition:** A family of pdfs or pmfs  $\{g(t; \theta); \theta \in \Theta\}$  for a univariate random variable  $T$  with real-valued parameter  $\theta$  has a monotone likelihood ratio (MLR) if, for every  $\theta_2 > \theta_1$ ,  $g(t, \theta_2)/g(t, \theta_1)$  is an increasing function of  $t$  on  $\{t : g(t, \theta_1) > 0 \text{ or } g(t, \theta_2) > 0\}$ .

## Monotone Likelihood Ratio (MLR)

**Theorem** Suppose that  $X$  has density  $f(x; \theta)$  with MLR in  $T(x)$ , then:

- 1 There exists a UMP level  $\alpha$  test of  $H_0 : \theta \leq \theta_0$  versus  $H_1 : \theta > \theta_0$  which is of the form  $C(x) = \{T(x) \geq k\}$
- 2  $\beta(\theta)$  is increasing in  $\theta$
- 3 For all  $\theta'$  this same test is the UMP level  $\alpha' = \beta(\theta')$  to test  $H_0 : \theta \leq \theta'$  versus  $H_1 : \theta > \theta'$
- 4 For all  $\theta < \theta_0$ , the test of (1) minimizes  $\beta(\theta)$  among all tests of size  $\alpha$



$N(\mu, \sigma^2)$ , with  $\sigma^2$  known

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu = \mu_1 \quad \mu_0 > \mu_1$$

$$R = \left\{ \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \leq -z_{1-\alpha} \right\}$$

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu < \mu_0$$

$$R = \left\{ \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \leq -z_{1-\alpha} \right\}$$

$$H_0 : \mu > \mu_0 \text{ vs } H_1 : \mu < \mu_0$$

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$N(\mu, \sigma^2)$ , with  $\sigma^2$  known

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu = \mu_1 \quad \mu_0 < \mu_1$$

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$H_0 : \mu = \mu_0$  vs  $H_1 : \mu \neq \mu_0$

$$R = \left\{ \left| \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \right| \geq z_{1-\alpha/2} \right\}$$

## $N(\mu, \sigma^2)$ , with $\sigma^2$ unknown

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$$R = \left\{ \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \leq -t_{1-\alpha}^{n-1} \right\}$$

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu < \mu_0$$

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## Method of inversion

One to one correspondence between tests and confidence intervals.

### Hypothesis testing:

Fix the parameter: asks what sample values (in the appropriate region) are consistent with that fixed value

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For each  $\theta_0 \in \Theta$ , let  $A(\theta_0)$  be the acceptance region of a level  $\alpha$  test  $H_0 : \theta = \theta_0$ . Define a set  $C(x) = \{\theta_0 : x \in A(\theta_0)\}$ . Then  $C(x)$  is a  $(1 - \alpha)$ -confidence set.



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# Hypothesis Testing and Confidence Intervals

In general, inverting acceptance region of a two sided test will give two sided interval and inverting acceptance region of a one sided test will give an open end interval on one side.

## Theorem

Let acceptance region of a two sided test be of the form  $A(\theta) = \{x : c_1(\theta) \leq T(x) \leq c_2(\theta)\}$  and let the cutoff be symmetric, that is,  $P_\theta(T(X) > c_2(\theta)) = \alpha/2$  and  $P_\theta(T(X) < c_1(\theta)) = \alpha/2$ . If  $T$  has MLR property then both  $c_1(\theta)$  and  $c_2(\theta)$  are increasing in  $\theta$ .