Exercise 1

Let $X_1, \ldots, X_n$ be iid from a population with p.d.f.:

$$f(x|\theta) = \frac{\theta}{x^2} \qquad x \geq \theta$$

Find $\hat{\theta}_{MLE}$.

Exercise 1

Joint Distribution

$$f(x_1, \ldots, x_n) = \prod_i \frac{\theta}{x_i^2} I(x_i \geq \theta)$$

Likelihood function $L(\theta)$

$$L(\theta | x_1, \ldots, x_n) = \frac{\theta^n}{\prod_i x_i^2} I(min\, x_i \geq \theta)$$

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1st derivative likelihood function

$$\frac{dL(\theta)}{d\theta} \quad = \quad \frac{n\theta^{n-1}}{\prod_i x_i^2} I(min\, x_i \geq \theta)$$

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MLE

$$\hat{\theta}_{MLE} = min(X_1, X_2, \ldots, X_n)$$

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MLE

$$\hat{\theta}_{MLE} = min(X_1, X_2, \ldots, X_n)$$

Exercise 4

Let X be distributed as a Poisson;

$$f(x;\lambda) = \frac{\lambda^x exp(-\lambda)}{x!}$$

Compare the following estimator for $\exp(-\lambda)$:

$$T_1 = exp\left(-\bar{X}\right) \qquad T_2 = \frac{\sum_{i=1}^{n} I(X_i = 0)}{n}$$

Exercise 4

$\hat{\lambda}_{MLE}$

$$\hat{\lambda}_{MLE} = \bar{X}$$
$$\hat{\lambda}_{MLE} \sim N\left(\lambda_0, \frac{\lambda_0}{n}\right)$$

Invariance Property of MLE

$$\theta = exp(-\lambda)$$
$$\hat{\theta}_{MLE} = exp(-\hat{\lambda}_{MLE})$$
$$\hat{\theta}_{MLE} = exp(-\bar{X})$$

Exercise 4

$\hat{\lambda}_{MLE}$

$$
\begin{aligned}
\hat{\lambda}_{MLE} &= \bar{X} \\
\hat{\lambda}_{MLE} &\sim N\left(\lambda_0, \frac{\lambda_0}{n}\right)
\end{aligned}
$$

Invariance Property of MLE

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\begin{aligned}
\theta &= exp(-\lambda) \\
\hat{\theta}_{MLE} &= exp(-\hat{\lambda}_{MLE}) \\
\hat{\theta}_{MLE} &= exp(-\bar{X})
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T_1 MLE Estimator of θ

Exercise 4

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T_1 MLE Estimator of θ

Exercise 4

Properties T_1

- asymptotically unbiased
- consistent
- asymptotically gaussian
- efficient

Var(T_1)? Delta Method

Let X a r.v. with $E(X) = \mu_X$, and $Var(X) = \sigma_X^2$. Theorem The following approximations hold:

$$E(g(X)) \approx g(\mu)$$

and

$$Var(g(X)) \approx [g'(\mu)]^2 \sigma_X^2$$

Exercise 4

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Exercise 4

Delta Method

$$
\begin{aligned}
Var(g(X)) &\approx [g'(\mu)]^2\sigma_X^2 \\
g(X) &= exp(-X) \\
g'(X) &= -exp(-X) \\
(g'(X))^2 &= exp(-2X) \\
X &= \bar{X} \\
\mu &= \lambda \\
\sigma^2 &= \frac{\lambda}{n}
\end{aligned}
$$

$Var(T_1)$

$$Var(T_1) = \frac{\lambda}{n} exp(-2\lambda)$$

Exercise 4

Delta Method

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Exercise 4

T_2

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\begin{aligned}
E(T_2) &= P(X_i = 0) = exp(-\lambda) \\
Var(T_2) &= \frac{exp(-\lambda)(1 - exp(-\lambda))}{n}
\end{aligned}
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Properties T_2

- unbiased
- consistent
- asymptotically gaussian (CLT)

Exercise 4

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- consistent
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Exercise 4

T_1 Better that T_2 beacuse is MLE

$var(T_1) \leq Var(T_2)$

$$
\begin{aligned}
\frac{\lambda exp(-2\lambda)}{n} &\leq \frac{exp(-\lambda)(1-exp(\lambda))}{n}\\
\lambda exp(-\lambda) &\leq (1-exp(\lambda))\\
exp(-\lambda) &\leq \frac{1}{1+\lambda}\\
exp(-\lambda) &\leq \frac{1}{1+\lambda}\\
-\lambda &\leq -log(1+\lambda)\\
\lambda &\geq log(1+\lambda)
\end{aligned}
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$$

Exercise 5

Let X be distributed according to a Laplace ;

$$f(x;b) = \frac{1}{2b} exp\left(-\frac{|x|}{b}\right) \qquad x \in R$$

Fine $\hat{b}_{MLE}$ and $\hat{b}_{MOM}$