Exercises 6^{th} Week

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Exercise 1

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as uniform $U[0, \theta]$

- Verify null hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ through likelihood ratio test, with level of significance fixed equal to α.
- Verify null hypothesis $H_0 : \theta = 7$ versus $H_1 : \theta = 8$ considering a test with the following rejection region:

  $$R = \{(x_1, x_2, \ldots, x_6) : max(X_1, X_2, X_3, X_4, X_5, X_6) > 6\}$$

  Calculate α and β

  - $$\alpha = P(X_{(n)} > 6 | \theta = 7) = 1 - \left(\frac{6}{7}\right)^6 = 0.6034$$
  - $$\beta = P(X_{(n)} < 6 | \theta = 8) = \left(\frac{6}{8}\right)^6 = 0.178$$

Exercise 2

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as

$$f(x; \theta) = \theta x^{\theta - 1} I_{(0,1)}(x) \qquad \theta > 0$$

1. Use the MLE of θ to find an approximate $100(1 - \alpha)\%$ confidence interval for θ.
2. Find the following statistics test
   - Likelihood ratio statistics test
   - Score test statistics
   - Wald test statistics

   Specifying the distributions of the previous test statistics

Exercise 3

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. Poisson random variables with mean θ

$$f(x;\theta) = \frac{\theta^x exp(-\theta)}{x!}$$

It is given that $\sum_{i=1}^{300} x_i = 1210$. Using likelihood ratio test, score test and Wald test to test the null hypothesis $H_0 : \theta = 4$, against the alternative hypothesis $H_1 : \theta \neq 4$ at level $\alpha = 0.05$, and compare the results obtained.

Observe that the joint pdf of $X = (X_1, \ldots, X_n)$ is give by

$$\begin{aligned}
f(x_1, \ldots, x : n; \lambda) &= \prod_i \frac{\lambda_i^x exp(-\lambda)}{x_i!} \\
L(\lambda) &= \frac{\lambda^{\sum_i x_i} exp(-n\lambda)}{\prod_i x_i!} \\
l(\lambda) &= \sum_i x_i log(\lambda) - n\lambda - log\left(\prod_i x_i!\right) \\
\frac{dl(\lambda)}{\lambda} &= \frac{\sum_i x_i}{\lambda} - n \\
\frac{d^2 l(\lambda)}{\lambda^2} &= -\frac{\sum_i x_i}{\lambda^2} \\
\hat{\lambda} &= \bar{x} = 5.5
\end{aligned}$$

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\frac{dl(\lambda)}{\lambda} &= \frac{\sum_i x_i}{\lambda} - n \\
\frac{d^2 l(\lambda)}{\lambda^2} &= -\frac{\sum_i x_i}{\lambda^2} \\
I_n(\lambda) &= -E\left(\frac{d^2 l(\lambda)}{\lambda^2}\right) = -\frac{n}{\lambda} \\
\hat{\lambda} &= \bar{x} = 4.03
\end{aligned}$$

- WALD Test statistics

  $$\frac{\bar{x} - \lambda_0}{\sqrt{\frac{\bar{x}}{n}}}$$

  approximate distribution $N(0,1)$

- $$\frac{\bar{x}-\lambda_0}{\sqrt{\frac{\bar{x}}{n}}} = \frac{4.03-4}{\sqrt{\frac{4.03}{300}}} = 0.26 < 1.96$$

  Accept null hypothesis

- Score test statistics

  $$\frac{\frac{\sum_i x_i}{\lambda_0} - n}{\sqrt{\frac{n}{\lambda_0}}}$$

  approximate distribution $N(0,1)$

  $$\frac{\frac{1210}{4} - 300}{\sqrt{\frac{300}{4}}} = 0.289$$

  Accept null hypotheisi

- Likelihood Ratio Test Statistics

$$\begin{aligned}
2 \times \left(l(\hat{\lambda}) - l(\lambda_0) \right) &= 2 \times \left(\sum_i x_i log \frac{\hat{\lambda}}{\lambda_0} - n(\hat{\lambda} - \lambda_0) \right) \\
\Lambda(x) &= 2 \times \left(\sum_i x_i log \frac{\bar{x}}{\lambda_0} - n(\bar{x} - \lambda_0) \right) \\
&= 0.0823 \\
p - value &= 0.23
\end{aligned}$$

LRT approximate distribution χ_1, Accept null hypothesis