Double Exponential

The double exponential distribution is

$$f(x|\theta) = \frac{1}{2}e^{-|x-\theta|}, \qquad -\infty < x < \infty$$

For an iid sample of size $n = 2m + 1$, show that the mle of θ is the median of the sample.

Double Exponential: Likelihood and log-likelihood

The double exponential distribution is

$$\begin{aligned} f(x_1,\dots,x_n|\theta) &= \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|} \\ L(\theta|x_1,\dots,x_n) &= \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^{n}|x_i-\theta|} \\ logL(\theta|x_1,\dots,x_n) &= -nlog(2) - \sum_{i=1}^{n}|x_i-\theta| \end{aligned}$$

Maximizing log-likelihood

$$g(\theta) = -\sum_{i=1}^{n} |x_i - \theta|$$

Note that $g(\theta)$ is a continuous function of θ and its derivative exists at all points θ that are not equal to any x_i

$$\begin{aligned}
g\prime(\theta) &= \frac{d}{d\theta} g(\theta) = -\sum_{i=1}^{n} (-1 \times 1(x_i > \theta) + (+1) \times 1(x_i < \theta)) \\
g\prime(\theta) &= \sum_{i=1}^{n} 1 \times 1(x_i > \theta) - \sum_{i=1}^{n} 1 \times 1(x_i < \theta) \\
g\prime(\theta) &= \begin{cases} positive & if \quad \theta < median(x_i) \\ negative & if \quad \theta > median(x_i) \end{cases}
\end{aligned}$$