

Gaussian distribution

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Sufficient Statistics

$$f(y_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, \dots, y_i, \dots, y_n | \mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

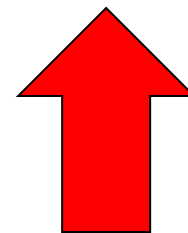
$$f(y_1, \dots, y_i, \dots, y_n | \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

σ^2 known, Sufficient Statistics for μ

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n y_i^2 - 2\mu \sum_{i=1}^n y_i + \sum_{i=1}^n \mu^2$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$



σ^2 known, Sufficient Statistics for μ

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

$$T(y) = \sum_{i=1}^n y_i \text{ sufficient statistics for } \mu$$

Sufficient Statistics for σ^2

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n y_i^2 - 2\mu \sum_{i=1}^n y_i + \sum_{i=1}^n \mu^2$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

Sufficient Statistics for σ^2

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n y_i^2 - 2\mu \sum_{i=1}^n y_i + \sum_{i=1}^n \mu^2$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

Sufficient Statistics for σ^2

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

$$T(y) = \left(\sum_{i=1}^n y_i, \sum_{i=1}^n y_i^2 \right) \text{ sufficient statistics for } \sigma^2$$

Sufficient Statistics for μ and σ^2

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$\sum_{i=1}^n (y_i - \mu)^2 = \sum_{i=1}^n y_i^2 - 2\mu \sum_{i=1}^n y_i + \sum_{i=1}^n \mu^2$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

Sufficient Statistics for μ and σ^2

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$f(y \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n y_i^2} e^{+\frac{1}{\sigma^2} \mu \sum_{i=1}^n y_i} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n \mu^2}$$

$$T(y) = \left(\sum_{i=1}^n y_i, \sum_{i=1}^n y_i^2 \right) \text{ sufficient statistics for } (\mu, \sigma^2)$$

Maximum Likelihood estimation

σ^2 known

$$f(y_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, \dots, y_i, \dots, y_n \mid \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma} \right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

Maximum Likelihood estimation

σ^2 known

$$L(\mu, \sigma^2 | y_i) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$l(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2$$

$$\frac{dl(\mu, \sigma^2)}{d\mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)$$

Maximum Likelihood estimation for μ and (σ^2 known)

$$0 = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)$$

$$\hat{\mu} = \bar{y}$$

$$\frac{d^2 l(\mu, \sigma^2)}{d\mu^2} = -\frac{n}{\sigma^2} < 0$$

Maximum Likelihood estimation

$$f(y_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, \dots, y_i, \dots, y_n | \mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2}$$

$$f(y_1, \dots, y_i, \dots, y_n | \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

Maximum Likelihood estimation

$$L(\mu, \sigma^2 | y_i) = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2}$$

$$l(\mu, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2$$

$$\frac{\partial l(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)$$

$$\frac{\partial l(\mu, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (y_i - \mu)^2$$

$u(\mu, \sigma^2)$ score function

$$\frac{\partial l(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)$$

$$\frac{\partial l(\mu, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (y_i - \mu)^2$$

Second Derivative

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial (\sigma^2)^2} = +\frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^n (y_i - \mu)^2$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} = -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \mu)$$

Second Derivative

$$E\left(\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu^2}\right) = -\frac{n}{\sigma^2}$$

$$E\left(\frac{\partial^2 l(\mu, \sigma^2)}{\partial (\sigma^2)^2}\right) = -\frac{n}{2\sigma^4}$$

$$E\left(\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu \partial \sigma^2}\right) = 0$$

$I(\mu, \sigma^2)$ Fisher Information

$$\begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix}$$

Second Derivative

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2}$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial (\sigma^2)^2} = +\frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^n (y_i - \mu)^2 = -\frac{n}{2\sigma^4}$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} = -\frac{1}{\sigma^4} \sum_{i=1}^n (y_i - \mu)$$

Maximum Likelihood estimation

$$0 = \frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \mu)$$

$$0 = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (y_i - \mu)^2$$

$$\hat{\mu} = \bar{y}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n}$$

Maximum Likelihood estimation

$$\hat{\mu} = \bar{y}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n}$$

Check Second Derivatives?

Second Derivative

$$\left. \frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu^2} \right|_{\hat{\mu}, \hat{\sigma}^2} = -\frac{n}{\hat{\sigma}^2}$$

$$\left. \frac{\partial^2 l(\mu, \sigma^2)}{\partial (\sigma^2)^2} \right|_{\hat{\mu}, \hat{\sigma}^2} = +\frac{n}{2\hat{\sigma}^4} - \frac{1}{\hat{\sigma}^6} \sum_{i=1}^n (y_i - \hat{\mu})^2 = -\frac{n}{2\hat{\sigma}^4}$$

$$\left. \frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} \right|_{\hat{\mu}, \hat{\sigma}^2} = -\frac{1}{\hat{\sigma}^4} \sum_{i=1}^n (y_i - \hat{\mu}) = 0$$

$$I(\mu, \sigma^2)^{-1}$$

Inverse of Fisher Information

$$\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

Asymptotic distribution mle

(μ_0, σ_0^2) real value of parameters

$$\sqrt{n} \left((\hat{\mu}, \hat{\sigma}^2) - (\mu_0, \sigma_0^2) \right) \rightarrow_{n \rightarrow \infty} N \left((0, 0), \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & 2\sigma_0^4 \end{bmatrix} \right)$$

$$(\hat{\mu}, \hat{\sigma}^2) \approx N \left((\mu_0, \sigma_0^2), \begin{bmatrix} \frac{\sigma_0^2}{n} & 0 \\ 0 & \frac{2\sigma_0^4}{n} \end{bmatrix} \right)$$