Exercises 1$^{\text{st}}$ Week — Solutions

Exercise 1

$X \sim N(8.4, 10^2)$, $n = 25$, so $\bar{X} \sim N\left(8.4, \dfrac{10^2}{25}\right) = N(8.4, 2^2)$,

a)

$$P(\bar{X} > 12) = P\left(Z > \frac{12 - 8.4}{2}\right) = P(Z > 1.8) = 1 - \Phi(1.8) \approx 0.0359$$

b)

$$P(\bar{X} < 10) = P\left(Z < \frac{10 - 8.4}{2}\right) = P(Z < 0.8) = \Phi(0.8) \approx 0.7881$$

Exercise 2

$X \sim U(5, 45)$, length $L = 40$, cdf $F(x) = \dfrac{x - 5}{40}$.

a)

$$P(X < 30) = \frac{30 - 5}{40} = 0.625$$

b) Sample $X_1, \ldots, X_5$ iid $U(5, 45)$ (the five working days).

i) The cdf of the maximum is $F_{(5)}(x) = [F(x)]^5$:

$$P(30 \leq X_{(n)} \leq 32) = \left(\frac{27}{40}\right)^5 - \left(\frac{25}{40}\right)^5 \approx 0.1401 - 0.0954 = 0.0447$$

ii) The cdf of the minimum is $F_{(1)}(x) = 1 - [1 - F(x)]^5$:

$P(10 \leq X_{(1)} \leq 12) = [1 - F(10)]^5 - [1 - F(12)]^5 = 0.875^5 - 0.825^5 \approx 0.5129 - 0.3822 = 0.1307$

Exercise 3

$X \sim \text{Exp}(\lambda)$ with mean 20, so $\lambda = 1/20$.

a)

$$P(X < 30) = 1 - e^{-30/20} = 1 - e^{-1.5} \approx 0.7769$$

b) By the memoryless property, the minimum of $n = 5$ iid $\text{Exp}(\lambda)$ random variables is itself exponential with rate $5\lambda = 0.25$:

$$P(10 \leq X_{(1)} \leq 12) = e^{-0.25 \cdot 10} - e^{-0.25 \cdot 12} = e^{-2.5} - e^{-3} \approx 0.08209 - 0.04979 = 0.0323$$

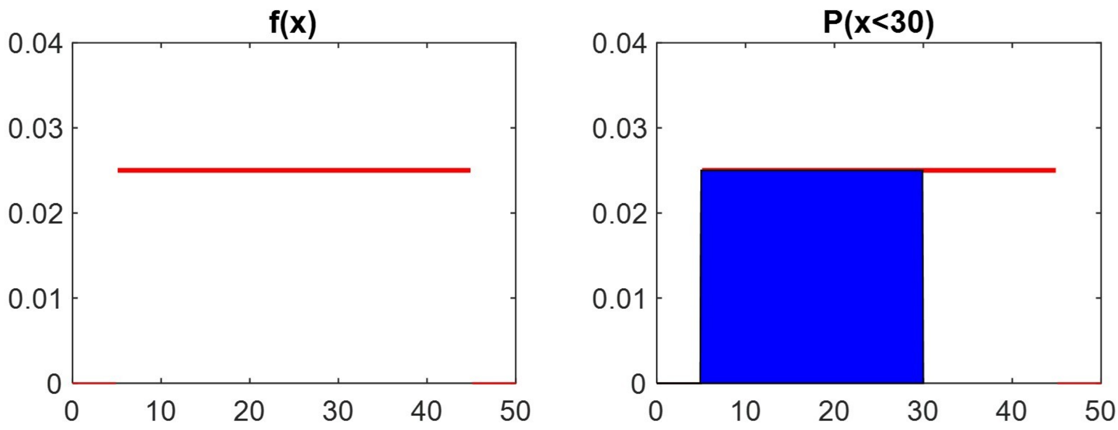

Figure 1: Uniform Distribution

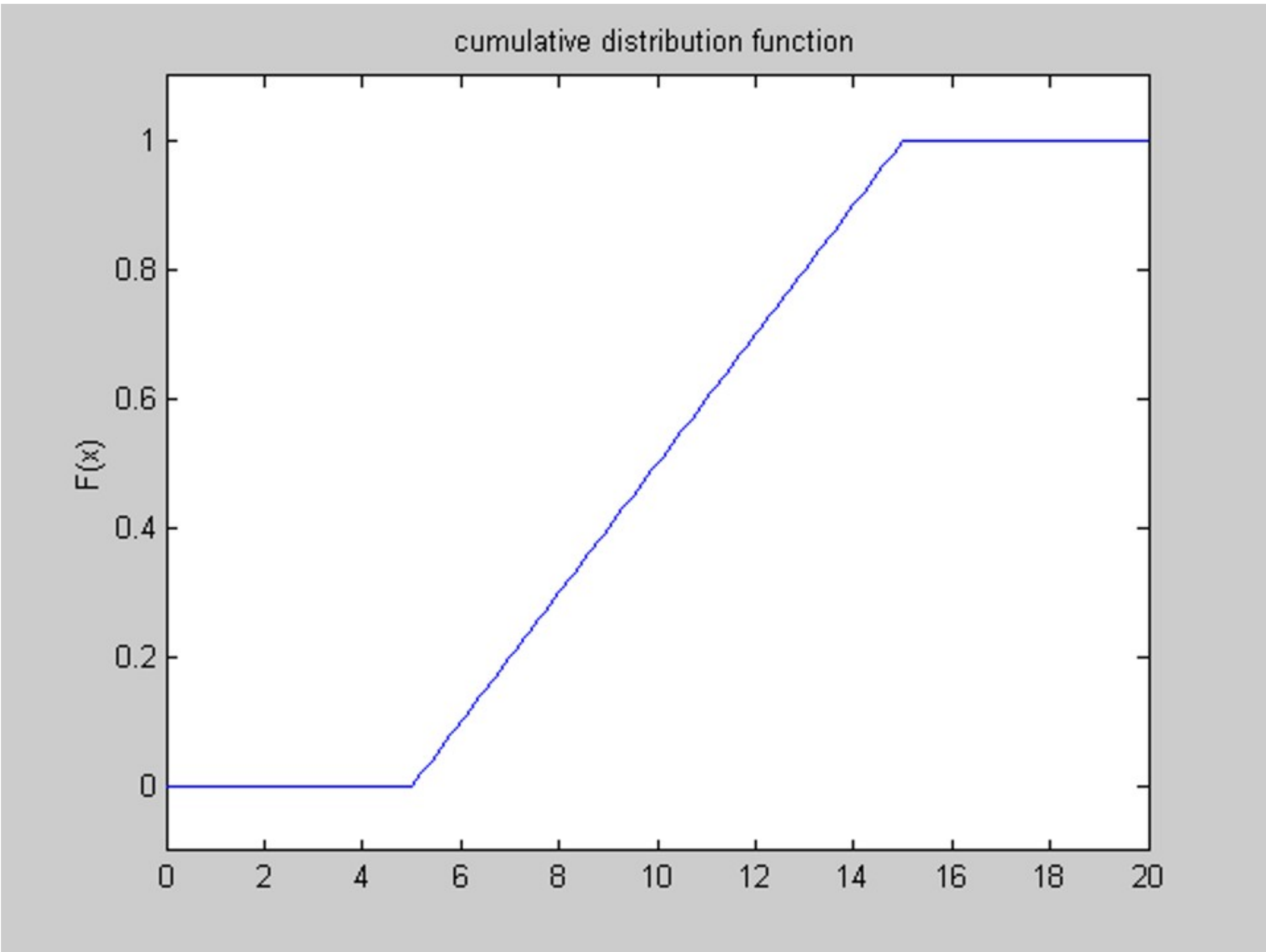

Figure 2: Uniform CDF

Exercise 4

$Y \sim \text{Bin}(n = 1700,\, p = 0.6)$: $\text{E}(Y) = 1020$, $\text{Var}(Y) = 1700 \cdot 0.6 \cdot 0.4 = 408$, $SD = \sqrt{408} \approx 20.20$.

We want $P(Y > 1060)$. Normal approximation **with continuity correction**:

$$P(Y > 1060) \approx P\left(Z > \frac{1060 - 1020}{20.20}\right) = P(Z > 1.98) \approx 0.024$$

(Without the continuity correction: $z = 1.98 \Rightarrow P \approx 0.0238$. The exact binomial value, computed numerically, is 0.02220 — the continuity-corrected normal approximation is closer to the true value, as expected.)

Exercise 5

$X \sim \text{Exp}(\beta)$, $\beta = 2$ (mean), hence rate $\lambda = 1/\beta = 0.5$; $n = 25$.

a)

$$\text{E}(\bar{X}) = \beta = 2, \qquad \text{Var}(\bar{X}) = \frac{\beta^2}{n} = \frac{4}{25} = 0.16$$

b) The sum $S_n = \sum_{i=1}^{n} X_i$ of n iid Exp(rate λ) random variables is a Gamma(shape $= n$, scale $= \beta$) random variable, here $\Gamma(25, 2)$. Since the scale is exactly 2, this is also a χ^2 **distribution with** $2n = 50$ **degrees of freedom**: $S_{25} \sim \chi^2_{50}$.

$$P(\bar{X} > 2.5) = P(S_{25} > 25 \times 2.5) = P(S_{25} > 62.5) = P(\chi^2_{50} > 62.5)$$

Computed numerically (Gamma/chi-squared):

$$P(\bar{X} > 2.5) \approx 0.11$$

c) CLT approximation: $\bar{X} \,\dot{\sim}\, N(2,\, 0.16)$, so

$$P(\bar{X} > 2.5) \approx P\left(Z > \frac{2.5 - 2}{0.4}\right) = P(Z > 1.25) \approx 0.106$$

Exercise 6

Let $U \sim U(0, 1)$ and $X = -\dfrac{1}{\lambda} \ln(1 - U)$ (with $\lambda > 0$). For $x > 0$:

$$\begin{aligned} P(X \le x) &= P\left(-\frac{1}{\lambda} \ln(1 - U) \le x\right) = P\big(\ln(1 - U) \ge -\lambda x\big) \\ &= P\big(1 - U \ge e^{-\lambda x}\big) = P\big(U \le 1 - e^{-\lambda x}\big) \end{aligned}$$

Since $x > 0 \Rightarrow 1 - e^{-\lambda x} \in (0, 1)$, and for $U \sim U(0, 1)$ we have $P(U \le u) = u$ for every $u \in [0, 1]$:

$$P(X \le x) = 1 - e^{-\lambda x}, \qquad x > 0$$

Moreover $X > 0$ almost surely (since $U \in (0, 1) \Rightarrow 1 - U \in (0, 1) \Rightarrow \ln(1 - U) < 0 \Rightarrow X > 0$), so $P(X \le x) = 0$ for $x \le 0$.

This is exactly the cdf of an Exp(λ) random variable: $F(x) = \big(1 - e^{-\lambda x}\big)\mathbb{1}(x > 0)$. Hence $X \sim \text{Exp}(\lambda)$. ■

(This is the inversion method: to generate exponential random variables from $U(0, 1)$ draws, one can use $X = -\frac{1}{\lambda} \ln(1 - U)$, or equivalently $X = -\frac{1}{\lambda} \ln U$, since U and $1 - U$ have the same $U(0, 1)$ distribution.)