Probability: Problem Set 1 Solutions

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1 Show that $A \perp B \Rightarrow A^C \perp B^C$.

Solution
We have to show that $A^C \perp B^C$, that is

$$\mathrm{P}(A^C \cap B^C) = \mathrm{P}(A^C)\,\mathrm{P}(B^C).$$

Left hand side:

$$\mathrm{P}(A^C \cap B^C) = \mathrm{P}(A \cup B)^C = 1 - \mathrm{P}(A \cup B) = 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B) =$$
$$= 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A)\,\mathrm{P}(B),$$

where in the first equality we use the De Morgan's law $(A \cup B)^c = A^c \cap B^c$, in the third equality we use the Law of Total Probability, and in the last one the hypothesis.
Right hand side:

$$\mathrm{P}(A^C)\,\mathrm{P}(B^C) = (1 - \mathrm{P}(A))(1 - \mathrm{P}(B)) = 1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A)\,\mathrm{P}(B),$$

therefore the equality is verified.

2 Define the events
A = ill
B = smoker,
and define the probabilities
$\mathrm{P}(B) = 0.4$
$\mathrm{P}(A \mid B) = 0.25$
$\mathrm{P}(A \mid B^C) = 0.07$.
What is the probability of being ill?
What is the probability of being smoker given that you are ill?

Solution

$$\mathrm{P}(A) = \mathrm{P}(A \cap B) + \mathrm{P}(A \cap B^C) = \mathrm{P}(A \mid B)\,\mathrm{P}(B) + \mathrm{P}(A \mid B^C)\,\mathrm{P}(B^C) =$$
$$= (0.25)(0.4) + (0.07)(0.6) = 0.142.$$

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{(0.25)(0.4)}{0.142} = 0.704.$$

3 Given a package with three balls, define X=number of broken balls in a package, and $p = 0.2$ the probability of a ball to be broken. (We are assuming that the fact that a ball is broken is independent on the state of the other balls). Which is the probability that the number of broken balls is less than one?

Solution

Since

$$X \sim B(3, 0.2),$$

we have that

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{3}{0} 0.2^0 (0.8)^3 + \binom{3}{1} 0.2^1 (0.8)^2 = 0.896.$$

4 28 people booked a flight. The probability that each passenger is coming at the check-in is 0.7. Which is the probability that more than 25 passenger come at the check-in? (We are assuming that each passenger is independent from the others).

Solution

Since

$$X \sim B(28, 0.7),$$

we have that

$$P(X \geq 25) = \sum_{x=25}^{28} \binom{28}{x} 0.7^x 0.3^{28-x} =$$

$$= \binom{28}{25} 0.7^{25} 0.3^3 + \binom{28}{26} 0.7^{26} 0.3^2 + \binom{28}{27} 0.7^{27} 0.3^1 + \binom{28}{28} 0.7^{28} 0.3^0 = 0.0157.$$

5 Compute the second moment and the variance of a r.v. T distributed ad a Geometric with parameter $p \in (0, 1]$.

Solution

If $T \sim Ge(p)$, then $P(T = n) = pq^{n-1}$, where $q = 1 - p$.

$$\mathrm{E}(T^2) = \sum_{n=1}^{\infty} n^2 P(T = n) = \sum_{n=1}^{\infty} n^2 pq^{n-1} =$$

$$= p\left(\sum_{n=1}^{\infty} n^2 q^{n-1}\right) = p\left(\sum_{n=1}^{\infty}(n^2-n)q^{n-1} + \underbrace{\sum_{n=1}^{\infty} n q^{n-1}}_{\text{first derivative of power series}}\right) =$$

$$= p\left(\sum_{n=1}^{\infty} n(n-1)q^{n-1} + \frac{1}{(1-q)^2}\right) = p\left(q\underbrace{\sum_{n=1}^{\infty} n(n-1)q^{n-2}}_{\text{second derivative of power series}} + \frac{1}{(1-q)^2}\right) =$$

$$= p\left(q\frac{2}{(1-q)^3} + \frac{1}{(1-q)^2}\right) = p\left(q\frac{2}{p^3} + \frac{1}{p^2}\right) = \frac{2q+p}{p^2} = \frac{q+1}{p^2}.$$

$$\mathrm{Var}(T) = \mathrm{E}(T^2) - (\mathrm{E}(T))^2 = \frac{q+1}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}.$$

6 Compute the second moment and the variance of a r.v. X distributed as a Poisson with parameter $\lambda > 0$.

Solution

If $X \sim \text{Poisson}(\lambda)$, then $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$.

$$\mathrm{E}(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k) = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\lambda^k}{(k-1)!} = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{k\lambda^{k-1}}{(k-1)!}.$$

Consider only the summation

$$\sum_{k=1}^{\infty}\frac{k\lambda^{k-1}}{(k-1)!} = \sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \frac{\lambda^{k-1}}{(k-1)!} =$$

$$= \sum_{k=2}^{\infty}\frac{\lambda^{k-1}}{(k-2)!} + \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} =$$

define $k - 1 = m$

$$= \lambda\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + \sum_{m=0}^{\infty}\frac{\lambda^m}{m!} =$$

define $k - 2 = j$

$$= \lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} + e^{\lambda} = e^{\lambda} + \lambda e^{\lambda} = e^{\lambda}[1+\lambda],$$

where we used the fact that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$. Therefore

$$\mathrm{E}(X^2) = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda}[1+\lambda] = \lambda(1+\lambda) = \lambda + \lambda^2.$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

7 Consider a random variable U with a density given by

$$f_U(x) = 2\frac{\ln x}{x} 1_{[1,c]}(x), \qquad c \geq 1.$$

a. Compute c.

b. Compute $\mathrm{E}(U^2)$ and $P(0 < U < 1)$.

Solution

a.
$$1 = \int_{\mathbb{R}} f_U(x)\mathrm{d}x = \int_{\mathbb{R}} 2\frac{\ln x}{x} 1_{[1,c]}(x)\mathrm{d}x = \int_1^c 2\frac{\ln x}{x}\mathrm{d}x =$$
$$= (\ln x)^2 \mid_1^c = (\ln c)^2 - (\ln 1)^2 = (\ln c)^2.$$
$$(\ln c)^2 = 1 \qquad \Rightarrow \qquad \ln c = \pm 1 \qquad \Rightarrow c = e,\ \frac{1}{e}.$$

Since $c \geq 1$, this implies $c = e$.

b.
$$\mathrm{E}(U^2) = \int_{\mathbb{R}} x^2 f_U(x)\mathrm{d}x = \int_1^e x^2 2\frac{\ln x}{x}\mathrm{d}x = \int_1^e 2x \log x \, \mathrm{d}x =$$
$$= x^2 \ln x \mid_1^e - \int_1^e \frac{x^2}{x}\mathrm{d}x = e^2 - \frac{x^2}{2} \mid_1^e = \frac{1}{2}(e^2+1).$$

Finally
$$P(0 < U < 1) = \int_0^1 f_U(x)\mathrm{d}x = \int_0^1 0 \, \mathrm{d}x = 0.$$

8 $V \sim \text{Poisson}(2)$. Order the following three numbers from the smallest to the biggest.

$$\frac{2}{9} \qquad 2F_V(0) \qquad P(|V-2| \geq 3)$$

Solution

Recall that if $X \sim \text{Poisson}(\lambda)$ then $\text{Var}(X) = \text{E}(X) = \lambda$.

Cebicev inequality says that

$$P(|V - \text{E}(V)| \geq a) \leq \frac{\text{Var}(V)}{a^2} \qquad\qquad \forall\, a > 0.$$

Therefore

$$P(|V-2| \geq 3) \leq \frac{2}{9}.$$

On the other hand

$$2F_V(0) = 2P(V \leq 0) = 2P(V = 0) = 2e^{-2}\frac{2^0}{0!} = \frac{2}{e^2} > \frac{2}{9}.$$

It follows that

$$P(|V-2| \geq 3) \leq \frac{2}{9} < 2F_V(0).$$

9 Let $X \sim N(0,1)$ and let Z be the Random Sign, namely Z is a r. v. with distribution given by

$$\text{P}(Z = 1) = \text{P}(Z = -1) = \frac{1}{2}.$$

Suppose that X and Z are independent and define $Y := ZX$. Prove that

a. $Y \sim N(0,1)$,

b. $X + Y$ is not gaussian.

Solution

a. We show that X and Y have the same distribution.

$$\text{P}(Y \in A) = \text{P}(ZX \in A) = \text{P}(Z = 1 \cap X \in A)\dot{\cup}\,\text{P}(Z = -1 \cap -X \in A) =$$

$$= \text{P}(Z = 1)\,\text{P}(X \in A) + \text{P}(Z = -1)\,\text{P}(-X \in A) =$$

$$= \frac{1}{2}\text{P}(X \in A) + \frac{1}{2}\text{P}(-X \in A) = P(X \in A),$$

where we used the fact that X and $-X$ have the same distribution.

b. $X + Y$ cannot be gaussian, indeed

$$\mathrm{P}(X + Y = 0) = \mathrm{P}(X + ZX = 0) = \mathrm{P}((1 + Z)X = 0) =$$
$$\mathrm{P}(1 + Z = 0) \cup \mathrm{P}(X = 0) =$$
$$= \mathrm{P}(1 + Z = 0) + \mathrm{P}(X = 0) - \mathrm{P}(1 + Z = 0) \cap \mathrm{P}(X = 0) = \frac{1}{2} + 0 - 0 = \frac{1}{2}.$$

10 Prove that

$$\mathrm{E}(X|Y) = \mathrm{E}(X) \Rightarrow \mathrm{Cov}(X, Y) = 0,$$

but not vice versa.

Solution

$$\mathrm{Cov}(X, Y) = \mathrm{E}(X, Y) - \mathrm{E}(X)\,\mathrm{E}(Y) = \mathrm{E}[\mathrm{E}(XY \mid Y)] - \mathrm{E}(X)\,\mathrm{E}(Y) =$$
$$= \mathrm{E}[Y\,\mathrm{E}(X \mid Y)] - \mathrm{E}(X)\,\mathrm{E}(Y) = \mathrm{E}(Y)\,\mathrm{E}(X) - \mathrm{E}(X)\,\mathrm{E}(Y) = 0,$$

where in the second equality we used the Law of Iterated Expectations, and in the fourth one the hypothesis.

To show that the vice versa is not true we use a counterexample.
Consider a random variable X such that $P(X = i) = \frac{1}{3}$, $i = -1, 0, 1$ and $X^3 = X$. Define $Y := X^2$. We have

$$\mathrm{E}(XY) = \mathrm{E}(X^3) = \mathrm{E}(X) = 0,$$

and

$$\mathrm{E}(X)\,\mathrm{E}(Y) = 0\,\mathrm{E}(Y) = 0,$$

therefore

$$\mathrm{Cov}(X, Y) = 0.$$

However

$$\mathrm{E}(Y|X) = \mathrm{E}(X^2|X) = X^2 \neq \mathrm{E}(Y),$$

because

$$\mathrm{E}(Y) = \mathrm{E}(X^2) = 0P(X = 0) + 1P(X = 1) + 1P(X = -1) =$$
$$= \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}.$$