INTEGRALS and EXPONENTIAL DENSITY FUNCTION

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we define the function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(Probability density function of an exponential distr.)

Since this is a DENSITY, its integral is equal to 1

[This concept will be very useful in PROBABILITY, STATISTICS and ECONOMETRICS... KEEP in mind!]

$$\int_0^{+\infty} \lambda e^{-\lambda x}\, dx = 1$$

You already prooved it in class, if you don't remember try to do it again by yourself

REMIND: substitute $\lambda x = y$
$\lambda dx = dy$

1st exercise

Compute the integral $\int_0^{+\infty} x \lambda e^{-\lambda x}\, dx$

NOTICE that this integral has the form $\int x f(x)\, dx$ where $f(x)$ is the density function discribed above

This kind of integralls will be called EXPECTED VALUE. We are not interested now in this concept, but it is better to get familiar with it!

Let's compute it!

$$\int_0^{+\infty} x \lambda e^{-\lambda x}\, dx =$$

multiply and divide by λ

$$= \frac{1}{\lambda}\int_0^{+\infty} \lambda x\, e^{-\lambda x} \lambda dx =$$

define $\lambda x = y$
$\lambda dx = dy$

substitute

[notice that this substitution does not imply any change on the domain of the interval because y goes from 0 to +∞]

$$= \frac{1}{\lambda}\int_0^{+\infty} y e^{-y}\, dy =$$

by the FUNDAMENTAL theorem of calculus, find the antiderivative of the function. Since we have a PRODUCT of functions, we must use the INTEGRATION BY PARTS (*)

$$= \frac{1}{\lambda}\left[-\frac{(1+y)}{e^y}\right]_0^{+\infty} =$$

compute the function at +∞ and 0

$$= \frac{1}{\lambda}\left[-0 - \left(-\frac{1}{1}\right)\right] = \frac{1}{\lambda} = E(X)$$

← EXPECTED VALUE of an exponentially distributed Random variable X

(*) INTEGRATION BY PARTS

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

In our case

$$\int_0^{+\infty} y\, e^{-y}\, dy = \qquad u(x) = y \rightarrow u'(x) = 1 \qquad v'(x) = e^{-y} \rightarrow v(x) = -e^{-y}$$

$$= y\left(-e^{-y}\right) - \int 1 \cdot \left(-e^{-y}\right) dy = -y e^{-y} - e^{-y} = -(1+y)\, e^{-y}$$

bianchetto!

$\boxed{2}$ $$\int_0^{+\infty} \lambda x^2 e^{-\lambda x}\, dx =$$

NOTICE that this integral is of the form $\int x^2 f(x)\, dx$ where $f(x)$ is the density distribution used above. You will call it SECOND MOMENT $E(X^2)$

multiply and divide by λ^2

$$= \frac{1}{\lambda^2} \int_0^{\infty} \lambda^2 x^2 e^{-\lambda x}\, \lambda\, dx =$$

define $\lambda x = y \quad \lambda dx = dy$

$$= \frac{1}{\lambda^2} \int_0^{\infty} y^2 e^{-y}\, dy =$$

INTEGRATION BY PARTS

bianchetto!

$$\int_0^{\infty} y^2 e^{-y}\, dy = y^2\left(-e^{-y}\right) - \int 2y\left(-e^{-y}\right) =$$

$$= -y^2 e^{-y} + 2 \int y\, e^{-y}\, dy$$

$$= \frac{1}{\lambda^2}\left(\left[-\frac{y^2}{e^y} \right]_0^{+\infty} + 2 \underbrace{\int_0^{+\infty} y\, e^{-y}\, dy}_{} \right) =$$

(*) previous exercise $\left[-(1+y)\, e^{-y}\right]_0^{\infty} = 1$

$$= \frac{1}{\lambda^2}\left((-0 + 0) + 2 \cdot 1 \right) = \frac{2}{\lambda^2} = E(X^2)$$

For future reference, consider that the variance of a random variable is defined as

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} = VAR(X)$$

in our case

$\leftarrow$ VARIANCE of an exponentially distributed RANDOM VARIABLES

DOMAIN

Find the domain of the following functions

I) $f(x,y) = \ln(1-x^2-y^2)$

II) $f(x,y) = \ln(x^2-1) + \ln(1-y^2)$

III) $f(x,y) = \ln\left(x \ln\left(\frac{1}{x+y}\right)\right)$

IV) $f(x,y) = \ln(x+y^2-1)$

V) $f(x,y) = \ln(x^2-y^2-1)$

SIGN

Determine the sign of f of the functions IV), V) and their level curves

find the domain, the sign and the level curve of the function

VI) $f(x,y) = \ln\left(\frac{x^2-4y^2-1}{16x^2+9y^2-1}\right)$, what if $f(x,y) = \sqrt{\frac{x^2-4y^2-1}{16x^2+9y^2-1}}$?

By yourself: $f(x_1,x_2) = \sqrt{\frac{4-(x_1^2+x_2^2)}{x_1-x_2}}$

$f(x_1,x_2) = \sqrt{\frac{2x_1-(x_1^2+x_2^2)}{x_1^2+x_2^2-x_1}}$

Hint: In order to recognize the curve described by each equation you have to make some algebra (add and subtract stuffs)

LEVEL CURVE

VII) $f(x,y) = \frac{x^2}{x^2+y^2}$

PARTIAL DERIVATIVES

VIII) $f(x,y) = (x+y)(x-y)$

IX) $f(x,y) = \frac{x}{x^2+y^2}$

X) $f(x,y) = \sqrt{x+2y}$

XI) $f(x,y) = \operatorname{sen}(xy) + \cos(xy)$

XII) $f(x,y) = \ln(x^2+y^2)$

XIII) $f(x,y) = \ln\sqrt{x^2+y^2}$

XIV) $f(x,y) = e^{x/y}$

XV) $f(x,y) = e^x/e^y$

XVI) $f(x,y) = \ln\frac{x+\sqrt{1+x^2}}{y+\sqrt{1+y^2}}$

XVII) $= f(x,y) = \ln(\operatorname{sen}(x^2+y^2))$

DOMAIN

I) $f(x,y) = \ln(1-x^2-y^2)$

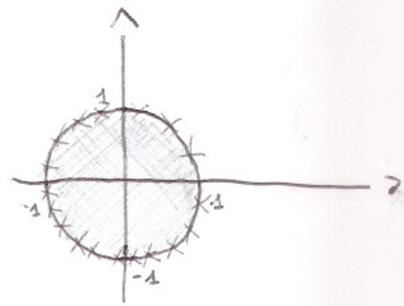
$\hookrightarrow D_f = \{(x,y) \in \mathbb{R}^2 \mid 1-x^2-y^2 > 0\}$

$x^2+y^2 < 1$ $\leftarrow$ which equation is it? $(x-a)^2+(y-b)^2=r^2$

$a=0$

$b=0$

$r=1$



$\leftarrow$ Why are we concidering the area ~~...~~ inside the circonference?

· Why are the boundaries not included?

· Is it an open or a closed set?

II) $f(x,y) = \ln(x^2-1) + \ln(1-y^2)$

We have to combine the two restrictions given by the two logarithms (They MUST HOLD together)

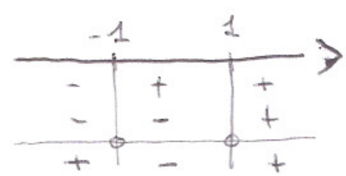
$D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2-1>0 \wedge 1-y^2>0\}$

$$\begin{cases} x^2-1>0 & (a) \\ 1-y^2>0 & (b) \end{cases}$$

(a) $x^2-1 = (x+1)(x-1) > 0$

$x+1>0, \quad x>-1$

$x-1>0, \quad x>1$

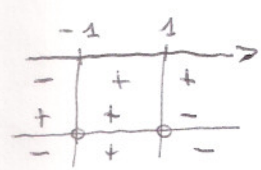


$x<-1 \vee x>1$

b) $1-y^2>0, \quad (1+y)(1-y)>0$

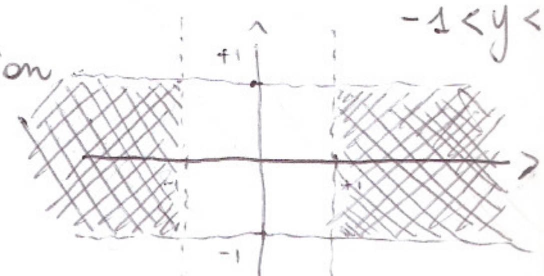
$1+y>0, \quad y>-1$

$1-y>0, \quad y<1$



$-1<y<1$

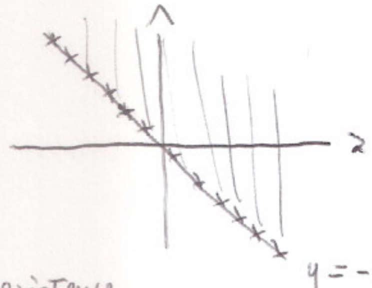
Intersection



III) $f(x,y) = \ln\left(x \ln\left(\frac{1}{x+y}\right)\right)$

. First of all, we have to find the conditions that guarantee the existence of the logarithm argument of the function

$\to \frac{1}{x+y} > 0 \quad x+y > 0 \quad y > -x$



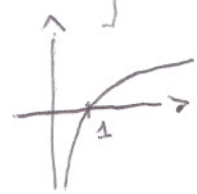
Now we study the condition that guarantees the existence of the function

$x \ln\left(\frac{1}{x+y}\right) > 0$

STUDY the sign of the product

$\begin{cases} x > 0 \\ \ln\left(\frac{1}{y+x}\right) > 0 \end{cases} \to$ How do you study when the $\ln(x) > 0$?

[This will be useful for the "SIGN" exercises]

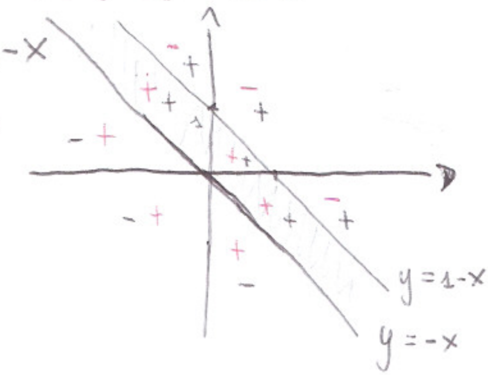


- OR you remember the shape of the logarithmic function $\ln(x) > 0$ if $x > 1$
- OR you remember the definition of logarithm $\ln x = y \quad e^y = x \to \ln(x) > 0,\ e^{\ln(x)} = x > e^0 = 1 \to x > 1$

From now on I'll only say that the argument of the logarithm must be greater than 1

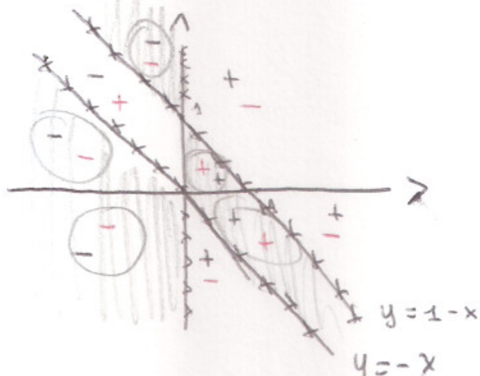
$\frac{1}{y+x} > 1, \quad \frac{1-y-x}{y+x} > 0$

- $1-y-x > 0, \quad y < 1-x$
- $y+x > 0, \quad y > -x$

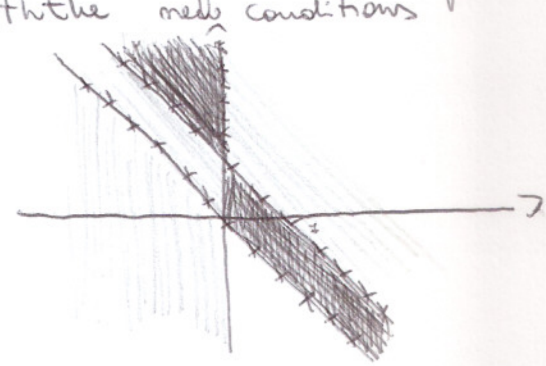


$\to$. $x > 0, \quad x > 0$

. $\ln\left(\frac{1}{y+x}\right) > 0,$ Shadow area



Last step: the condition for the existence of the inner logarithm MUST hold together with the new conditions



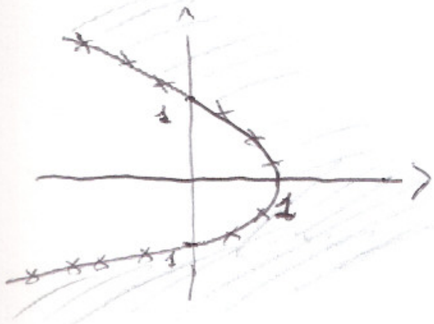
The darker shadowed area

IV) $f(x,y) = \ln(x+y^2-1)$

$D_f = \{(x,y) \in \mathbb{R}^2 \mid x+y^2-1 > 0\}$

$x+y^2-1 > 0 \rightarrow x > 1-y^2$

PARABOLA with horizontal axis of symmetry



$x=0 \quad y=\pm 1$

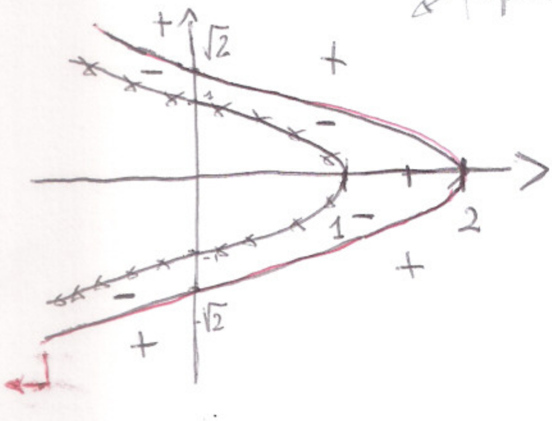
Now we study the SIGN of the function (given the DOMAIN, when the function assumes positive or negative values)

Since we are studying a log. function, we follow the reasoning in the previous exercise

Sorry for the proportions!

$\ln(x+y^2-1) > 0$ if $x+y^2-1 > 1$

$x+y^2-2 > 0 \quad x > 2-y^2$



$L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

Zero level curve

V) $f(x,y) = \ln(x^2-y^2-1)$

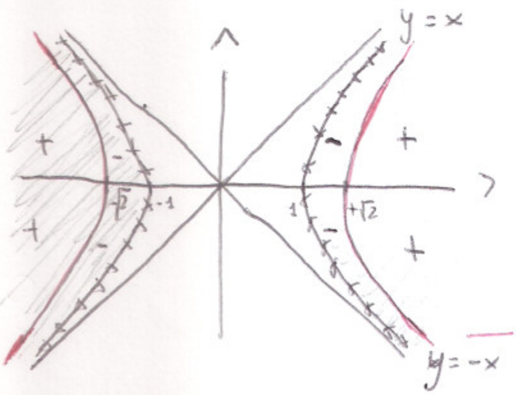
$D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2-y^2-1 > 0\}$

$x=0$

$y=0 \quad x^2=1 \quad x=\pm 1$

$x^2-y^2=1$ hyperbola

↳ Find the ASYMPTOTES (corresponding degenerate hyperbola $x^2-y^2=0$) $\rightarrow$ $x=y$, $x=-y$



→ 0 level curve

SIGN of the function

$\ln(x^2-y^2-1) > 0$ if $x^2-y^2-1 > 1$

$x^2-y^2 > 2$

$\frac{x^2}{(\sqrt{2})^2} - \frac{y^2}{(\sqrt{2})^2} > 1$ → the asymptotes are the same as before

$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = 0 \quad \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} = 0$

$\frac{x^2}{(\sqrt{2})^2} = 1$

$x = \pm\sqrt{2}$

VI) $f(x,y) = \ln\left(\frac{x^2-4y^2-1}{16x^2+9y^2-1}\right)$

$$D_f = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2-4y^2-1}{16x^2+9y^2-1} > 0\right\}$$

REMIND

To study the domain and to study the sign of the function are two different analysis!!

$N: x^2-4y^2-1>0 \rightarrow$ HYPERBOLA

$D: 16x^2+9y^2-1>0 \rightarrow$ ELLIPSIS

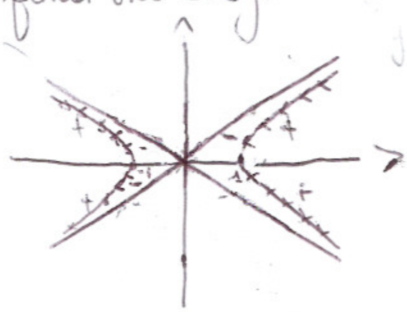
N: find the asymtotes $x^2-4y^2=0$, $(x+2y)(x-2y)=0$

$x=-2y$, $x=2y$

$y = \frac{x}{2}$

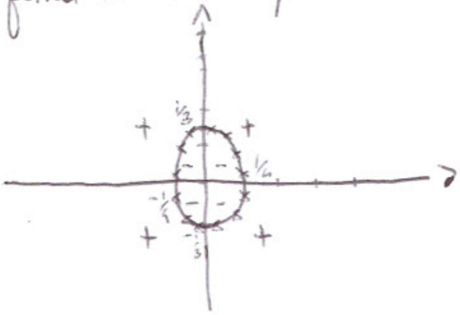
$x=0 \quad \cancel{-4y^2=1}$

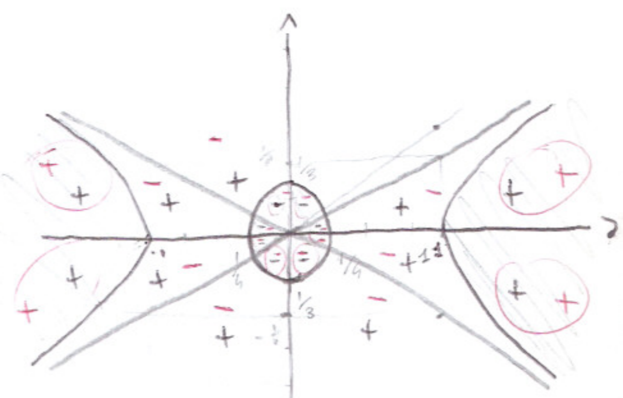
$y=0 \quad x^2=1, x=\pm 1$



D: find the intercept with the axis: $x=0, y=\pm\frac{1}{3}$

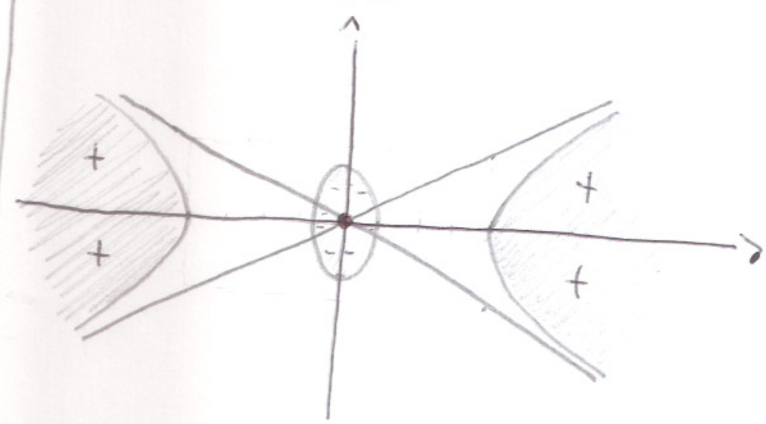
$y=0, x=\pm\frac{1}{4}$





* the FUNCTION is POSITIVE when the Denominator is POSITIVE (and $(x,y) \neq (0,0)$)

Now we have to intersect this condition with the DOMAIN



the origin is the zero level curve ($f(x,y)=0$)

STUDY the sign of the function (argument >1)

$$\frac{x^2-4y^2-1-16x^2-9y^2+1}{16x^2+9y^2-1} > 0 \qquad \frac{-15x^2-13y^2}{16x^2+9y^2-1} > 0$$

$\frac{N}{D} > 1 \quad \frac{N}{D} - 1 > 0$

We already studied the denominator

we need to study the numerator (ellipsis)

$$15x^2+13y^2 < 0 \quad \forall (x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)$$

*

VII) $f(x,y) = \dfrac{x^2}{x^2+y^2}$

$D_f = \left\{ (x,y) \in \mathbb{R}^2 \mid \exists!\, z \in \mathbb{R} : z = \dfrac{x^2}{x^2+y^2} \right\}$

$D_f = \{ \text{"} \mid x^2+y^2 \neq 0 \} = \mathbb{R}^2 \setminus \{(0,0)\}$

(What can you say about the sign?)

LEVEL CURVES $L_c(f) = \left\{ (x,y) \in \mathbb{R}^2 \mid \dfrac{x^2}{x^2+y^2} = c \right\}$

If $c<0 \rightarrow L_c(f) = \emptyset$ (the function never assumes negative values)

If $c=0 \rightarrow L_c(f) = \{ (x,y) \in \mathbb{R}^2 \mid x_1 = 0 \}$

If $0<c\leq 1 \rightarrow x^2 = c(x^2+y^2)$

$x^2 = cx^2 + cy^2$

$(1-c)x^2 = cy^2$

$y^2 = \dfrac{1-c}{c} x^2 \rightarrow y = \pm\sqrt{\dfrac{1-c}{c}}\, x^2$ lines passing through $(0,0)$

If $c>1 \rightarrow \dfrac{1-c}{c} < 0 \Rightarrow L_c(f) = \emptyset$

DIFFERENTIATION RULES

- linearity of differentiation $h(x) = af(x) + bg(x)$
  $h'(x) = af'(x) + bg'(x)$
- product of two functions $h(x) = f(x)g(x)$
  $h'(x) = f'(x)g(x) + f(x)g'(x)$
- chain rule (function of a function) $h(x) = f(g(x))$
  $h'(x) = f'(g(x))\, g'(x)$
- power rule $f(x) = x^m$
  $f'(x) = mx^{m-1}$
- quotient rule $h(x) = \dfrac{f(x)}{g(x)};\ h'(x) = \dfrac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
- functional power rule $h(x) = f^{g(x)} = e^{g(x)\ln f}$
  $h'(x) = f^{g(x)}\left(f'(x)\dfrac{g(x)}{f(x)} + g'(x)\ln f \right)$
- exponential / logarithmic $h(x) = c^{ax}$ $h'(x) = c^{ax}\ln c \cdot a \Rightarrow h(x) = e^x;\ h'(x) = e^x$
  $h(x) = \log_c x$ $h'(x) = \dfrac{1}{x\ln c} \Rightarrow h(x) = \ln x;\ h'(x) = \dfrac{1}{x}$