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Optimization
Problem Set 8 - Solutions

- **Exercise 1**

  Calculate

  $$\int_E xy\,\mathrm{d}x\mathrm{d}y$$

  where

  $$E = \{(x,y)) \in \mathbb{R}^2 : 0 < x < 1,\ x^2 < y < \sqrt{x}\}$$

  Solution: $\int_E xy\,\mathrm{d}x\mathrm{d}y = \frac{1}{12}$

- **Exercise 2**

  Maximize the function

  $$f(x,y) = x + y$$

  subject to

  $$x^2 + y^2 = 1$$

  Solution: Max in $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$; min in $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

- **Exercise 3**

  Compute the max/min of the following functions, given the constraints.

  $$f(x,y) = x^2 + 3y$$

  s.t. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

  Solution

  Define the function $g(x,y)$
  $g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$

  First of all, draw the function g. This is an ellipsis with intercept of the axis y at ± 3 (closed, bounded therefore compact, Weierstrass theorem holds)

Write the Lagrangian

$$\mathbb{L}(x,y,\lambda) = f(x,y) - \lambda\, g(x,y) = x^2 + 3y - \lambda\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right)$$

$$\mathbb{L}_x = 2x - \lambda\frac{x}{2} = 0$$
$$\mathbb{L}_y = 3 - \lambda\frac{2y}{9} = 0$$
$$\mathbb{L}_\lambda = -\frac{x^2}{4} - \frac{y^2}{9} + 1 = 0$$

With parametrization

$$\begin{cases} x = x_0 + a\cos\theta \\ y = y_0 + b\,\sin\theta \end{cases}$$

with $\theta \in [0, 2\pi)$

$$\begin{cases} x = 2\cos\theta \\ y = 3\,\sin\theta \end{cases}$$

$$f(\theta) = 4\cos^2\theta + 9\sin\theta = 4(1-\sin^2\theta) + 9\sin\theta = -4\sin^2\theta + 9\sin\theta + 4$$

$$f'(\theta) = -8\sin\theta\cos\theta + 9\cos\theta = \cos\theta(9 - 8\sin\theta) = 0$$

- $\cos\theta = 0$, therefore $\theta = \frac{\pi}{2}$, $\theta = \frac{3}{2}\pi$
- $9 - 8\sin\theta = 0$, therefore $\sin\theta = \frac{9}{8}$ since it is > 1, there are no solutions. $9 - 8\sin\theta > 0 \forall\theta$

$$f'(\theta) > 0 \qquad \text{iff} \qquad \cos\theta > 0$$

Therefore, at $\theta = \frac{\pi}{2}$ max, $\frac{3}{2}\pi$ min.

$$\begin{cases} x = 2\cos\dfrac{\pi}{2} = 0 \\ y = 3\sin\dfrac{\pi}{2} = 3 \end{cases}$$

- **Exercise 4**

  Consider the three vectors

$$v_1 = \begin{pmatrix} c \\ c+2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} c \\ 2 \\ c \end{pmatrix} \qquad v_3 = \begin{pmatrix} c \\ 1 \\ 1 \end{pmatrix}$$

  1) For which values of the parameter c do they form a basis of $\mathbb{R}^3$?

  2) Let A the 3×3 matrix whose columns are given by v_1, v_2, v_3 for the values of β satisfying the condition of the point 1). Solve the linear equation system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  using: i) the Cramer rule; ii) the substitution-elimination techniques.

  Solution

  1) $c \neq -1, 0, 1$.

  2) $(x, y, z) = \left(\frac{1}{1-c^2}, \frac{1}{c-1}, \frac{c}{1-c^2}\right)$

- **Exercise 5**

  Compute the eigenvalues and the associated eigenvectors of the following matrices:

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} ; B = \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}$$

  Solution

  Eigenvalues of A: $\lambda = 1$ with eigenspace $V_1 =$ vectors of the form: $(\alpha, -\alpha)$; $\lambda = 6$ with eigenspace $V_6 =$ vectors of the form: $(4\beta, \beta)$.

  Eigenvalues of B: $\lambda = 0$ with eigenspace $V_0 =$ vectors of the form: $(-2\alpha, \alpha)$; $\lambda = 21$ with eigenspace $V_{21} =$ vectors of the form: $(3\beta, \beta)$.

- **Exercise 6**

  Fix the parameter h so that the matrix

$$C = \begin{pmatrix} h & 1 & 0 \\ 1-h & 0 & 2 \\ 1 & 1 & h \end{pmatrix}$$

has the eigenvalue 1.

Solution

$h = 2$