

Mathematics
Academic year 2026-2027
Teacher: P. Gibilisco

Lecture 1 – Monday, August 31, 2026 (14:00-16:30)

The beginner ... should not be discouraged if ... he finds that he does not have the prerequisites for reading the prerequisites.
P.Halmos

Introduction to the course.

An example on the applications of mathematics in economics: risk aversion and concavity.

List of the notions required in advance to follow the course.

Deterministic and random phenomena.

The "law of chance" as an oxymoron.

Axiomatization of geometry and probability: Euclid and Kolmogorov.

The ideas of Kolmogorov: events are represented by sets; probability is a normed measure on them.

Probability spaces and their properties (finitely additive case).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

No proposition is in itself either probable or improbable, just as no place can be intrinsically distant ...
J.M.Keynes

Conditional probability, pairs of independent events ($A \perp\!\!\!\perp B$ means A and B are independent). Remark: $P(A) = P(A|\Omega)$.

The two pillars of probability theory: on one side analysis and measure theory; on the other side gambling situations, coin-tossing.

Lecture 2 – Tuesday, September 1, 2026 (14:00-16:30)

*... the fact is that there is
nothing as dreamy and poetic,
nothing as radical, subversive,
and psychedelic, as mathematics
... Mathematics is the purest of
the arts, as well as the most
misunderstood.
P.Lockhart*

Exercise: prove that $A \perp\!\!\!\perp B \Rightarrow A \perp\!\!\!\perp B^c$

Bayes formula.

Partitions.

Exercise: an urn contains w white balls and r red balls. Choose a ball in the urn and leave it out (without looking at its colour). Choose a second ball. Which is the probability that the second ball is white? (The answer will explain the queue paradox).

Pairwise independence and independence.

A counterexample: pairwise independence does not imply independence.

Exercise: prove that

$$p(A|B) + p(A^c|B) = 1,$$

while (in general)

$$p(A|B) + p(A|B^c) \neq 1.$$

(Conditional probability is a probability measure only with respect to the first argument!)

Lecture 3 – Wednesday, September 2, 2026 (14:00-16:30)

The geometric series and its sum.

$$1 + q + q^2 + q^3 + \cdots = \frac{1}{1 - q} \quad \text{if} \quad |q| < 1$$

Taylor polynomial and Taylor series. Example:

$$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k \quad (|x| < 1)$$

Taylor polynomial and Taylor series. Example:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \in \mathbb{R})$$

*No one shall expel us from the
paradise which Cantor has
created for us.
D. Hilbert*

Countable and uncountable sets. Cardinality of $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$.

Counting subsets of a set (n choose k). Newton binomial formula. The cardinality of the power set: $2^n = \sum_{k=0}^n \binom{n}{k}$.

Lecture 4 – Thursday, September 3, 2026 (14:00-16:30)

Counting measure on $\mathbb{N}$ is not continuous.

σ -algebras and σ -additivity for a probability measure.

Probability spaces and their properties (σ -additive case).

Continuity of a probability measure. Equivalence of continuity from above and σ -additivity (no proof!)

Coin flipping and the Bernoulli process. The probability to have k successes flipping a coin n times:

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Image of a function.

Random variables: discrete case

Distribution of a random variable.

Exercise. A fair die is rolled two times. Let us denote by X the first result and by Y the second result. Let $U := X - Y$. Which is the distribution of the random variable U ?

Lecture 5 – Friday, September 4, 2026 (11:00-13:30)

Non-negative series with sum equal to one and discrete distributions.

The most important discrete r.v.: Bernoulli, Binomial, Geometric and Poisson distribution.

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{c}{n}\right)^n = e^c$$

Convergence of the Bernoulli to the Poisson.

Independence for discrete random variables.

Expected value for discrete r.v.

Linearity and positivity for the expected value.

How the Bernoulli, Binomial, Geometric and Poisson distributions arise flipping a coin (Bernoulli process).

Mean value for the Bernoulli and binomial distribution.

A counterexample. Consider a box where there are three balls with the numbers $-1, 0, 1$. Choose randomly a ball and let X be the random variable which represent the result so that:

$$P(X = -1) = P(X = 0) = P(X = 1) = \frac{1}{3}.$$

Prove that $X \not\perp X^2$.

Lecture 6 – Monday, September 7, 2026 (14:00-16:30)

$$\mathbb{E}(g(X)) = \sum_k g(x_k)P(X = x_k)$$

Moments of a r.v.. Variance.

Covariance and its properties.

Standard deviation.

Correlation coefficient and scale invariance.

$$X \perp Y \quad \Rightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

$$X \perp Y \quad \Rightarrow \quad \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

$$X \perp Y \quad \Rightarrow \quad \text{Cov}(X, Y) = 0$$

A counterexample. Consider a box where there are three balls with the numbers $-1, 0, 1$. Choose randomly a ball and let X be the random variable which represent the result so that:

$$P(X = -1) = P(X = 0) = P(X = 1) = \frac{1}{3}.$$

Prove that $X \not\perp X^2$ while $\text{Cov}(X, X^2) = 0$ (this is true in general for X and X^2 when X has a distribution which is symmetric w.r.t. to 0).

This prove that:

$$X \perp Y \quad \not\Rightarrow \quad \text{Cov}(X, Y) = 0$$

Variance for the Bernoulli and Binomial distribution.

The derivative of the geometric series.

Waiting times for the Bernoulli process and the geometric distribution.

Expectation of the geometric distribution.

Lecture 7 – Tuesday, September 8, 2026 (14:00-16:30)

Cumulative distribution function.

Continuous random variables ($P(X = x) = 0 \forall x \in \mathbb{R}$, continuity of F_X).

Densities and absolutely continuous random variable.

The uniform density.

Exponential density.

Moments and variance for absolutely continuous random variables.

Mean value and variance for the uniform distribution.

Mean value and variance for the exponential distribution.

Cumulative distribution function for the uniform distribution.

If X is an a.c. r.v. with density f_X then

$$F'_X = f_X$$

Lecture 8 – Wednesday, September 9, 2026 (14:00-16:30)

A mathematician is someone to whom the facts that

$$2 + 2 = 4$$

and

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

are equally obvious.
Lord Kelvin

Cumulative distribution function for the exponential distribution.

The Gaussian density

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Functions whose antiderivative cannot be expressed as an elementary function.

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi} \quad (\text{No proof})$$

Density of $aX + b$ from the density of X .

$$\int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad \text{Integration by parts}$$

Mean and variance for the gaussian density.

The Gaussian density: the general case ($X \sim \mathcal{N}(\mu, \sigma^2)$ means $X = \sigma Z + \mu$ where $Z \sim \mathcal{N}(0, 1)$.)

Lecture 9 – Thursday, September 10, 2026 (14:00-16:30)

Standardized random variables

$$X^* := \frac{X - \mathbb{E}(X)}{\sigma(X)}$$

- The law of large numbers. (Distribution form; no proof yet). $X_1, \dots, X_n, \dots$ i.i.d. random variables with finite mean μ and variance σ^2 . Then

$$\frac{S_n}{n} \longrightarrow \mu \quad \text{in law}$$

- CLT. (No proof yet). $X_1, \dots, X_n, \dots$ i.i.d. random variables with finite mean and variance. Then

$$S_n^* \longrightarrow \mathcal{N}(0, 1) \quad \text{in law}$$

*Written in five years, may it last
as many thousands.
G. Cardano at the end of Ars
Magna*

Mac Laurin series for the trigonometric functions.

Introduction to complex numbers.

Exercise: calculate

$$\operatorname{Re} \left(\frac{1}{2 + 3i} \right) = \dots$$

History of $0, 1, i, e, \pi$.

Complex exponential as a power series.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

The Euler formula.

$$e^{i\pi} + 1 = 0$$

The Fundamental theorem of Algebra.

Complex random variables and their expectations.

A constant random variable is independent from any random variable.

Remark: $X \perp Y$ implies $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$.

The characteristic function, first properties.

- $\varphi_X(0) = 1$
- $\varphi_{\lambda X}(t) = \varphi_X(\lambda t)$
- $X \perp Y \quad \Rightarrow \quad \varphi_{X+Y}(t) = \varphi_X(t) \varphi_Y(t)$

Lecture 10 – Tuesday, September 15, 2026 (14:00-16:30)

$e^{i\pi} + 1 = 0$ is for boys,
 $i^i = \frac{1}{\sqrt{e^\pi}}$ is for men.
Anonymous

The inversion theorem (no proof!):

X, Y have the same distribution iff $\varphi_X(t) = \varphi_Y(t) \quad \forall t \in \mathbb{R}$

$$X \sim \text{Poisson}(\lambda) \implies \varphi_X(t) = \exp(\lambda(e^{it} - 1))$$

Using the characteristic function prove that

$$X \perp Y, \quad X \sim \text{Poisson}(\lambda), \quad Y \sim \text{Poisson}(\mu) \implies X + Y \sim \text{Poisson}(\lambda + \mu)$$

If $X_n \sim B(n, p)$ then $\varphi_X(t) = (pe^{it} + q)^n$. Using this result prove that the sum of independent Bernoulli variables is binomial.

- Convergence in law (in distribution).
- Example: if X_n and X are random variables whose values are natural numbers then convergence in law is equivalent to (no proof)

$$P(X_n = k) \rightarrow P(X = k) \quad \forall k \in \mathbb{N}$$

- Example: if $X_n \sim B(n, \frac{\lambda}{n})$ and $X \sim \text{Poisson}(\lambda)$ then $X_n \implies X$ in law.
- Continuity theorem (no proof!):

$$X_n \implies X \text{ in law} \quad \text{iff} \quad \varphi_{X_n}(t) \rightarrow \varphi_X(t) \quad \forall t \in \mathbb{R}$$

- Example: prove that if $X_n \sim B(n, \frac{\lambda}{n})$ and $X \sim \text{Poisson}(\lambda)$ then $X_n \implies X$ in law using the characteristic function and the continuity theorem.

Lecture 11 – Thursday, September 17, 2026 (14:00-16:30)

Students don't need a perfect teacher. Students need a happy teacher, who's gonna make them excited to come to school and grow a love for learning.
R. Feynman

The Rest of the Taylor polynomials in x_0 is *negligible near* x_0 .
Partial derivatives. Derivation under the integral sign.
Moments and the characteristic function.

$$\varphi_X^{(k)}(0) = i^k \mathbb{E}(X^k)$$

The Taylor formula for $\varphi_X(\cdot)$.

The characteristic functions of a constant random variable $X = c$: $\phi_X(t) = e^{ict}$.

Characteristic function of the gaussian: if $X \sim \mathcal{N}(0, 1)$ then $\phi_X(t) = e^{-\frac{t^2}{2}}$. (No proof!)

- LLN. $X_1, \dots, X_n, \dots$ i.i.d. random variables with finite mean μ and variance σ^2 . Then

$$\frac{S_n}{n} \longrightarrow \mu \quad \text{in law}$$

- CLT. $X_1, \dots, X_n, \dots$ i.i.d. random variables with finite mean μ and variance σ^2 . Then

$$S_n^* \longrightarrow \mathcal{N}(0, 1) \quad \text{in law}$$

- Meaning of the CLT: the sum of (infinitely) many, small, independent effects is (approximately) normal (gaussian).
- CLT reformulated. Let $\Phi(t)$ the c.d.f. of the standard gaussian distribution.

$X_1, \dots, X_n, \dots$ i.i.d. random variables with finite mean μ and variance σ^2 . Then

$$P(X_1 + \dots + X_n \leq x) = P\left(S_n^* \leq \frac{x - n\mu}{\sigma\sqrt{n}}\right) \approx \Phi\left(\frac{x - n\mu}{\sigma\sqrt{n}}\right)$$

Characteristic functions of an arbitrary Gaussian. The sum of independent Gaussian is Gaussian.

Lecture 12 – Friday, September 18, 2026 (11:00-13:30)

Simulation 1. – Quiz 7. Exercise 4.

Simulation 2. – Quiz 2.

Simulation 3. – Quiz 1, 6, 7.

Simulation 4. – Quiz 1, 6. Exercise 4.

Lecture 13 – Monday, September 21, 2026 (14:00-16:30)

The Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx \quad \alpha > 0$$

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad \Gamma(1) = 1 \quad \Gamma(n) = (n-1)!$$

$X \sim \Gamma(\alpha, \lambda)$ (where $\alpha, \lambda > 0$) if

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \cdot 1_{(0, \infty)}(x)$$

If

$$c \cdot x^{\alpha-1} e^{-\lambda x} \cdot 1_{(0, \infty)}(x)$$

is a density then

$$c = \frac{\lambda^\alpha}{\Gamma(\alpha)}$$

If X has density f_X then X^2 has density

$$f_{X^2}(x) = \frac{1}{2\sqrt{x}} (f(\sqrt{x}) + f(\sqrt{-x})) \cdot 1_{(0, \infty)}(x)$$

If $X \sim \mathcal{N}(0, 1)$ then $X^2 \sim \Gamma(\frac{1}{2}, \frac{1}{2})$.

This implies

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- Moments of the Gamma distribution. If $X \sim \Gamma(\alpha, \lambda)$ then

$$\mathbb{E}(X^n) = \frac{1}{\lambda^n} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}$$

$$\text{Var}(X) = \frac{\alpha}{\lambda^2}$$

If $X \sim \Gamma(\alpha, \lambda)$ then

- The characteristic function (no proof):

$$\varphi_X(t) = \left(\frac{\lambda}{\lambda - it} \right)^\alpha$$

- If $X \sim \Gamma(\alpha, \lambda)$, $Y \sim \Gamma(\beta, \lambda)$ and $X \perp Y$ then $X + Y \sim \Gamma(\alpha + \beta, \lambda)$.
- The chi-squared distribution with n degrees of freedom. If $X \sim \Gamma(\frac{n}{2}, \frac{1}{2})$ then we say that $X \sim \chi^2(n)$.
- If $X_1, X_2, \dots, X_n$ are independent standard gaussian then $X_1^2 + X_2^2 + \dots + X_n^2 \sim \chi^2(n)$.

Counterexample: S random sign, $X \sim \mathcal{N}(0, 1)$, $S \perp X$ then $Y = SX \sim \mathcal{N}(0, 1)$. Moreover $P(X + Y = 0) = \frac{1}{2}$ therefore $X + Y$ is not Gaussian.

Lecture 14 – Tuesday, September 22, 2026 (14:00-16:30)

Real vector spaces and scalar products.

Examples: $\mathbb{R}^n$, real polynomials, continuous functions on the unit interval, random variables on a probability spaces.

Exercise: prove that $E(X)$ is the constant "nearest" to the random variable X w.r.t. the distance associated to scalar product $\langle X, Y \rangle = E(XY)$.

Cauchy-Schwartz inequality in the plane, for functions, for random variables. Application: $|\rho(X, Y)| \leq 1$.