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Financial Market Models

Lecture 1

One-period model of financial markets & hedging problems

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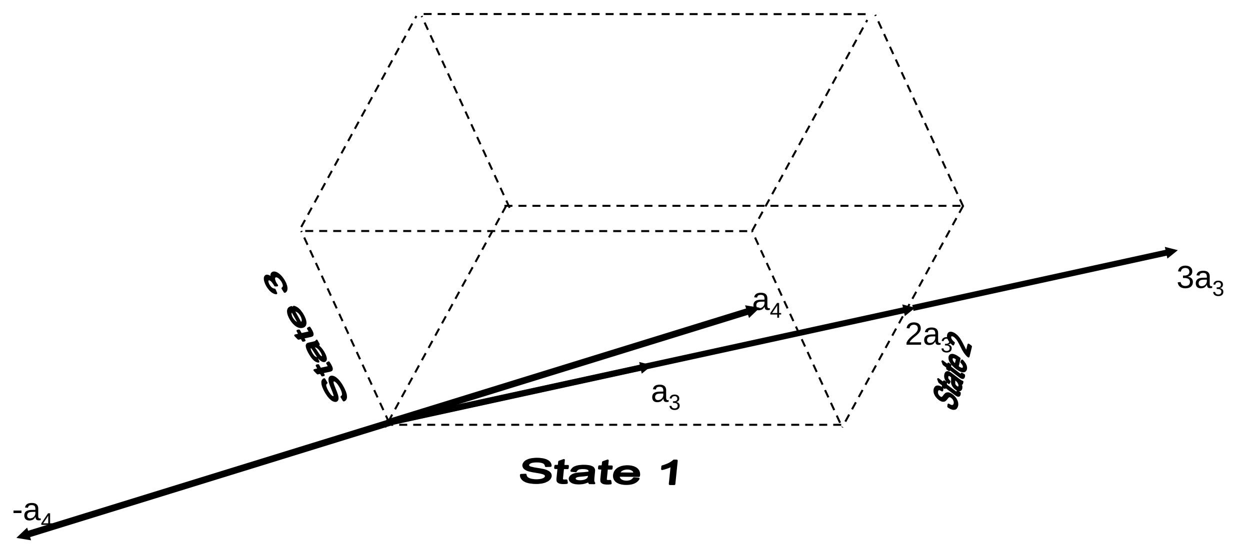
One-period model of financial markets



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Aims of section

- Introduce one-period model with finite number of states and the basic asset-pricing terminology
- Formulate hedging problem and try to solve it
- Explain the concepts of complete and incomplete markets
- Explain the role of matrix inverse in hedging and pricing
- Introduce “state prices” and discuss two different methods for asset pricing:

  1) by replication
  2) using a pricing kernel

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Introductory example

- There are four assets available

1) Uncertain stock price – 3 scenarios
   - Stock value tomorrow = 3, 2, or 1 with probability 1/2, 1/6,1/3
2) Risk-free asset with value 1 tomorrow
3) Two derivative securities – options on the stock

   3.1) Call option #1 struck at K = 1.5

   3.2) Call option #2 struck at K = 1

- Task: sell option #2, and to reduce risk exposure construct a hedging portfolio consisting of stock and risk-free asset

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Uncertainty in one-period model

- Two dates: today and tomorrow
- Value of all securities known today
- The tomorrow’s payoffs are uncertain
- Organization of uncertainty:
  – Finite number of scenarios
  – Each scenario known in detail today
  – Probability of each scenario known today

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Model asset payoffs

- Model stock and derivative payoffs

Probability	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
Stock	3	2	1
Call option #1 ($K = 1.5$)	1.5	0.5	0
Call option #2 ($K = 1$)	2	1	0

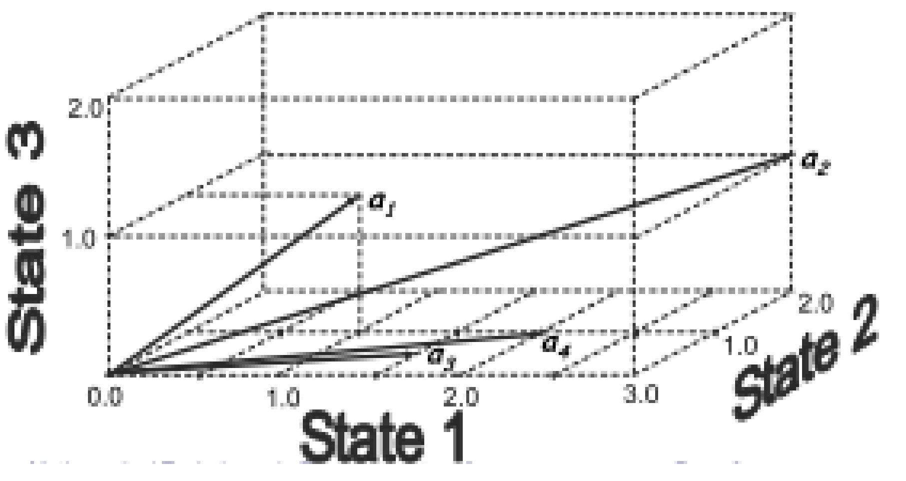
Payoffs as vectors

- Risk free asset : $a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- Stock : $a_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
- Options : $a_3 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}, a_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Graphical representation of the payoffs

- Payoffs in state-space form

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix} \quad a_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$



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Create vectors in Matlab

$a1 = \mathrm{ones}(3,1)$ or $a1=[1;1;1];$

$a2 = [3;2;1];$

$a3 = [1.5;0.5;0];$

$a4 = [2;1;0];$

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Operation on securities/vectors

- Scalar multiplication: Leverage
  - Example: Buy two units of option #1

$$2a_3 = 2\begin{bmatrix}1.5\\0.5\\0\end{bmatrix} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

  - Matlab command: $2 * a_3$

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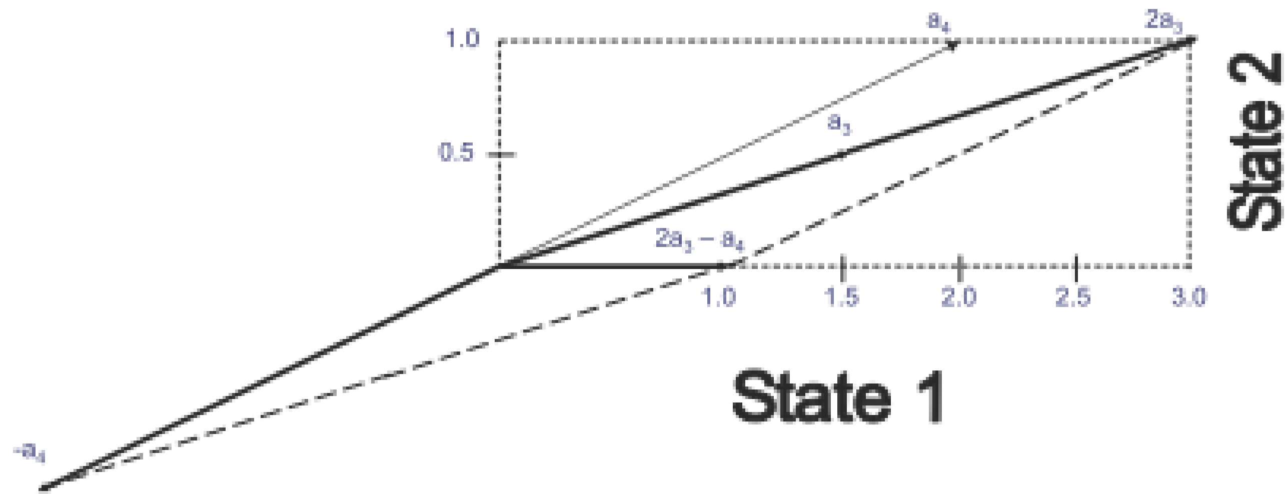
Operation on securities/vectors (cont.)

- Addition: Portfolios
  - Example: Buy two units of option #1, sell one unit of option #2

$$2a_3 - a_4 = \begin{bmatrix} 2*1.5-2 \\ 2*0.5-1 \\ 2*0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Matlab command : $2*a3 - a4$



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Matrix as a collection of securities/vectors

- It is common to work with several vectors at once and it is natural to form a matrix

Vectors: $a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $a_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $a_3 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}$ $a_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

A matrix: $\begin{bmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

- Matlab command: A = [a1 a2 a3 a4]

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Matrix operation: Transposition

- Sometimes we need a *row vector* rather than a *column vector* .
- Achieved by *transposition* of a column vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x^* = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

- Conversely, transposition of a *row vector* gives a *column vector*

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Matrix notation

- Matrix notation: $M \in \mathbb{R}^{m \times n}$
  - Where m is the number of rows and n is the number of columns
- The element in the *i*-th row and *j*-th column is denoted M_{ij}
- The entire *j*-th column is denoted $M_{.j}$
- The entire *i*-th row is denoted $M_{i.}$

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Matrix notation (cont.)

- $M \in \mathbb{R}^{m \times n}$

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix} = \begin{bmatrix} M_{1\cdot} \\ M_{2\cdot} \\ \vdots \\ M_{m\cdot} \end{bmatrix} = \begin{bmatrix} M_{\cdot 1} & M_{\cdot 2} & \cdots & M_{\cdot n} \end{bmatrix}$$

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Matrix operation: Transposition

- $M \equiv M_{ij} \longrightarrow M^* \equiv M_{ji}$
- $M \in \mathbb{R}^{m \times n} \longrightarrow M^* \in \mathbb{R}^{n \times m}$

Example

$$1)\quad x = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix},\ x^* =$$

$$2)\quad A = \begin{bmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{bmatrix},\ A^* =$$

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Matrix operation: Multiplication

- $U \in \mathbb{R}^{m \times k}$ and $V \in \mathbb{R}^{k \times n} \longrightarrow UV \in \mathbb{R}^{m \times n}$

$$\text{UV}=\begin{bmatrix} U_{1\cdot} \\ U_{2\cdot} \\ \vdots \\ U_{m\cdot} \end{bmatrix} \begin{bmatrix} V_{\cdot 1} & V_{\cdot 2} & \dots & V_{\cdot n} \end{bmatrix} = \begin{bmatrix} U_{1\cdot}V_{\cdot 1} & U_{1\cdot}V_{\cdot 2} & \dots & U_{1\cdot}V_{\cdot n} \\ U_{2\cdot}V_{\cdot 1} & U_{2\cdot}V_{\cdot 2} & \dots & U_{2\cdot}V_{\cdot n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{m\cdot}V_{\cdot 1} & U_{m\cdot}V_{\cdot 2} & \dots & U_{m\cdot}V_{\cdot n} \end{bmatrix}$$

- Example

$$U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad UV = ?$$

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Properties of matrix multiplication

- Not commutative

$$UV \neq VU$$

- Associative

$$(UV)W = U(VW)$$

- Transposition reverses order of multiplication !

$$(UV)^* = V^*U^*$$

Matrix and its application to portfolios

- Example: Suppose we sell 2 units of option #1, one unit of option #2, buy two units of stocks and borrow one unit of bond. What are the payoffs of this portfolio ?

Given:

$$bond = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, stock = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, option\#1 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix} \text{ and option\#2} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

- Assuming prices of securities are $S_{bond} = 1, S_{stock} = 3, S_{option\#1} = 1, S_{option\#2} = 2$, what is the price of this portfolio?

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Hedging Problem

- $Ax = b$
  - A denotes payoff matrix of basis assets
  - x denotes portfolio of basis assets
  - b denotes payoff of a *focus asset*
- Bank wants to issue a security b
- To offset risk, it separately buys hedging portfolio x
- The risk is reduced to $Ax - b$
  - If $Ax - b = 0$, then the bank's position is perfectly hedged
- The bank will price the security at S^*x + overheads and risk premium.

Hedging Problem (cont.)

- Example:

Given:

$$b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, a1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, a2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, a3 = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}$$

Your client wants to buy a security with payoff b, knowing there are assets $a1, a2, and\ a3$ traded in the markets. What is the strategy that gives you a perfect hedge? And how much you would charge the client if $S_{a1} = \frac{1}{1.05}\ and S_{a2} = 2$?

Hedging Problem (cont.)

Solution:

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Hedging Complications

- If inverse of A exists: $x = A^{-1}b$ is a perfect hedge and the solution is also unique

- Now consider $\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}$
  – No solution, Inverse does not exist
- Consider $\begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
  - Solution exists, inverse does not

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Linear independence and redundant assets

- Some complications in the hedging problem are caused by redundant assets

- An asset is redundant if it can be replicated by other assets

- Securities $A_{\cdot 1}, \ldots, A_{\cdot n}$ are linearly independent if none of them is a portfolio payoff of the remaining n-1 securities

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Dimension of marketed subspace

- Marketed subspace = all portfolio payoffs Ax generated by all possible basis assets x
  - Mathematically $Span(A_{\cdot 1}, \dots, A_{\cdot n})$
- Each linearly independent asset adds a new dimension to the marketed subspace
- Maximum number of lin. ind. assets = dimension of marketed subspace
- Dimensionality Theorem: The lin. ind. assets can be chosen in many ways, but their number is always the same = dimension

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Hedging in a complete market without redundant basis assets

- Complete market $\Leftrightarrow$

  # of linearly independent basis assets = # of states

- **Theorem:**

Suppose we have m states and a complete market with m basis assets. Then the payoff matrix is invertible and the hedging portfolio for any focus asset b is given by

$$x = A^{-1}b$$

Suppose prices of basis assets are stored in vector S. Under frictionless trading the only possible price of the focus asset equals

$$S^{*}x = S^{*}A^{-1}b$$

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Find dimension of marketed subspace

- Concepts
  - Suppose $A_{\cdot 1}, \dots, A_{\cdot k}$ are linearly independent. Then either $A_{\cdot k+1}$ is redundant or $A_{\cdot 1}, \dots, A_{\cdot k+1}$ are linearly independent
  - With *m* states, there cannot be more than *m* linearly independent assets.
- How we do it
  - Sort securities into two baskets
    - Linearly independent
    - Redundant

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Example

- Which of the following assets are linearly independent and redundant assets? Is this market complete?

$$A_{\cdot 1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A_{\cdot 2} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, A_{\cdot 3} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}, A_{\cdot 4} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

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Dimension and matrix rank

- Maximum number of linearly independent columns in a matrix A is called rank, *r(A)*
- If A is a payoff matrix of basis assets then *r(A)* = dimension of marketed subspace
- Facts about rank
  - $r(A^*A) = r(A)$
  - $r(AB) \leq \min\big(r(A), r(B)\big)$
  - $r(A) = r(A^*)$
  - If A is $m \times n$ then $r(A) \leq \min(m, n)$
- When $r(A) = \min(m, n)$, we say *A* has full rank
- Square matrices with full rank are invertible (non-singular matrices)

Identity matrix and Arrow-Debreu securities

- Arrow-Debreu (elementary) security for state *j*, denoted e_j, has payoff 1 in state *j* and payoff zero in all other states

- With m scenarios stacking all elementary securities into a matrix gives an m x m identity matrix

Matrix Inverse

- For every square matrix A with full rank there is a matrix B such that $AB = BA = I$

- Matrix B is unique, it is called the inverse to matrix A and it is commonly denoted by A^{-1}

- Facts:
  - If C, D are invertible then $(CD)^{-1} = D^{-1}C^{-1}$
  - $(A^{-1})^{-1} = A$

Inverse matrix and hedging portfolios

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- Interpretation of inverse matrix
- Let split $AA^{-1} = I$ by columns: $AA^{-1}_{\bullet j} = e_j$
- This looks like $Ax = e_j$
- Therefore $j - th$ column of A^{-1} gives replicating portfolio to Arrow-Debreu security e_j
- Example: if $A = \begin{bmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{bmatrix}$ and knowing that $A^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ -1 & 3 & -2 \\ 2 & -4 & 2 \end{bmatrix}$, show that Arrow-Debreu security e_j can be replicated by portfolio $A^{-1}_{\bullet j}$

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State prices

- State price *j* (ψ_j) is the price of an Arrow-Debreu security e_j
- Vector ψ is called the “state price vector”
- Assuming complete market and no redundant assets
  - State price vector

$$\psi^* = S^* A^{-1},$$

where S is the vector of prices of the basis assets and A is the matrix of the payoffs of the basis assets

Proof :

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Pricing formulae

- Suppose A is invertible (square, full rank)
- Complete market, no redundant basis assets
- Perfect hedge $Ax = b$
- Two ways to find the price of focus asset *b*

1. By replication:

   Focus asset price = $\boldsymbol{S^{*}x}$

2. Using state price (pricing kernel):

   Focus asset price = $\boldsymbol{\psi^{*}b}$

Proof of the second formula:

Example

- What is the price of $b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ given basis assets $A = \begin{bmatrix} 1 & 3 & 1.5 \\ 1 & 2 & 0.5 \\ 1 & 1 & 0 \end{bmatrix}$ with prices S= $\begin{bmatrix} 1 \\ 2 \\ 0.6 \end{bmatrix}$
  - By replication:
  - By state price:

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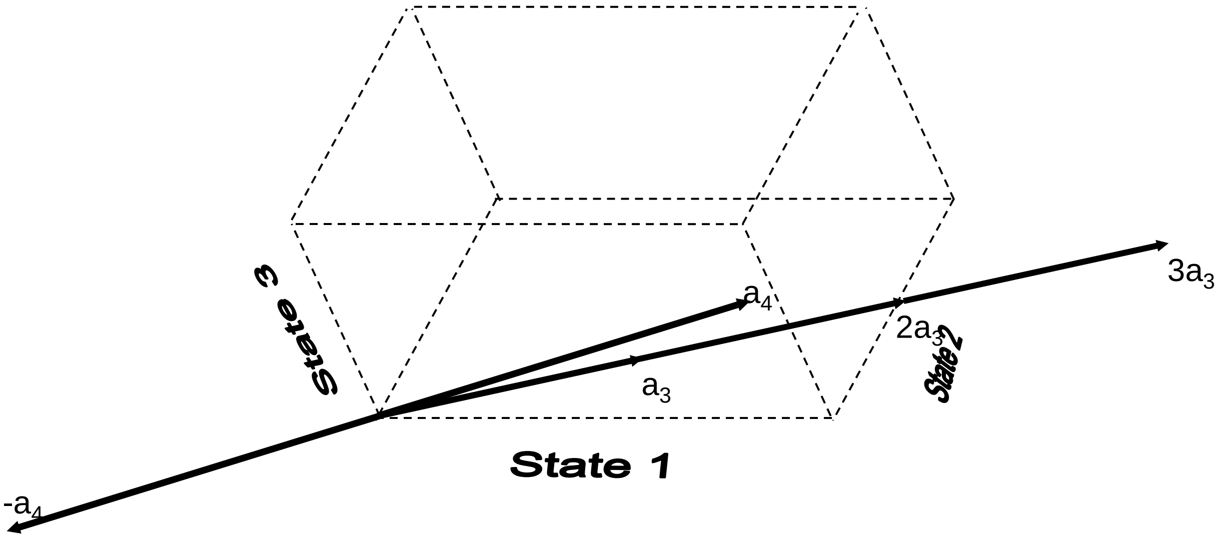
Hedging in an incomplete market



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Aims of this section

- Explain how to compute replicating portfolios in an incomplete market

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Hedging formula for incomplete markets, no redundant basis assets

- $r(A) = n < m$, fewer assets than states

$$Ax = b \qquad (1)$$

- multiply by A^* from the left

$$A^*Ax = A^*b \qquad (2)$$

- A^*A is square with full rank, it has an inverse

$$x = (A^*A)^{-1} A^*b \qquad (3)$$

- x solves (2) but does it solve (1)?
- Hedging error = $Ax\text{-}b = A(A^*A)^{-1} A^*b\text{-}b$
- If hedging error = 0 then solution exists and is given by (3), otherwise there is no solution

Example: Incomplete markets without redundant basis assets

- Is there a perfect hedge of a focus asset b considering that we can trade portfolio A ?

$$A = [A_{\bullet 1}\ A_{\bullet 2}] = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad b = A_{\bullet 3} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0 \end{bmatrix}$$

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Hedging problem with redundant basis assets

- Redundant basis assets do not affect existence of a solution, they merely add free parameters to the solution
- $A = A_1 \mid A_2$, $A_2 = A_1C$, $r(A)=r(A_1)$, $x = x_1|x_2$
  - If A_1 is square, market is complete
    $$x_1 = (A_1)^{-1}b - Cx_2$$
  - If A_1 is not square, market is not complete
    $$x_1 = (A_1^*A_1)^{-1}A_1^*b - Cx_2$$
  - Hedging error $= A_1 (A_1^*A_1)^{-1}A_1^*b - b$
  - x_2 represents the free parameters in the solution
- Example 2.3 pp. 28-29

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The Least-Squares Hedge

- In practice most markets are incomplete $r(A) = n < m$, fewer assets than number of states

- This means we cannot always find a perfect hedge (replication) for a focus asset $b \longrightarrow$ mathematically, $Ax = b$ cannot always be solved.

- Instead, we would like to find the best *approximate hedge* according to some criterion

The Least-Squares Hedge

- Replication error : $\varepsilon = Ax - b$
- Criterion: Minimize the Sum of Squared Replication Errors

$$SSRE = {\varepsilon_1}^2 + {\varepsilon_2}^2 + \cdots + {\varepsilon_m}^2$$

- Optimal hedge = least-squares hedging portfolio

$$\boldsymbol{x} = (\boldsymbol{A}^*\boldsymbol{A})^{-\mathbf{1}}\boldsymbol{A}^*\boldsymbol{b}$$

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Least Squares Hedge – Example

- Focus asset $b^{*}= [1\ 2\ 3]$
- Basis assets: $A_{.1}{}^{*} = [1\ 1\ 0]\ and\ \ A_{.2}{}^{*} = [0\ 1\ 0]$

What is the least-squares hedge of the focus asset?

Solution:

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Least squares hedge - Geometry

- Optimal criteria
  - Minimize $\varepsilon^*\varepsilon$

(square length of vector ε)

- Point of minimal distance

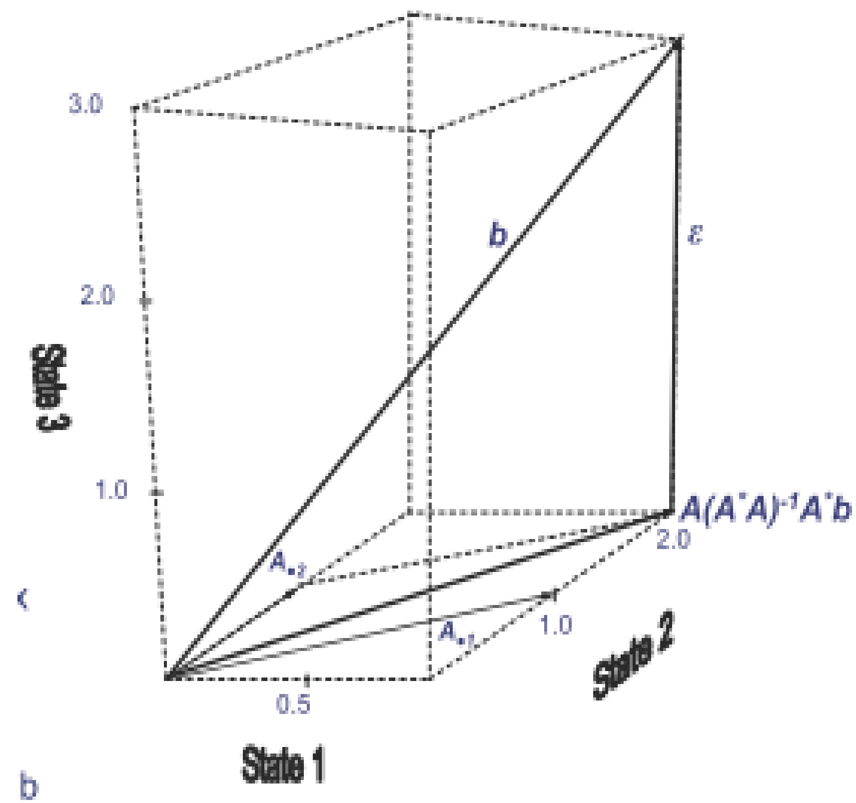
of Ax from b:

- ε must be orthogonal to

all vectors in marketed subspace A

- Mathematically $A^*\varepsilon = 0$
- $A^*(b - Ax) = 0 \implies$

$$\boldsymbol{x} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}\boldsymbol{A}^*\boldsymbol{b}$$



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Minimizing the Expected Squared Replication Error (ESRE)

- Not all scenarios are equally likely: we need criterion putting more weight on likely scenarios:

$$ESRE = p_1{\varepsilon_1}^2 + p_2{\varepsilon_2}^2 + \cdots + p_m{\varepsilon_m}^2$$

- The optimal hedge is :

$$\widehat{\boldsymbol{x}} = (\widetilde{\boldsymbol{A}}^*\widetilde{\boldsymbol{A}})^{-1}\widetilde{\boldsymbol{A}}^*\widetilde{\boldsymbol{b}}$$

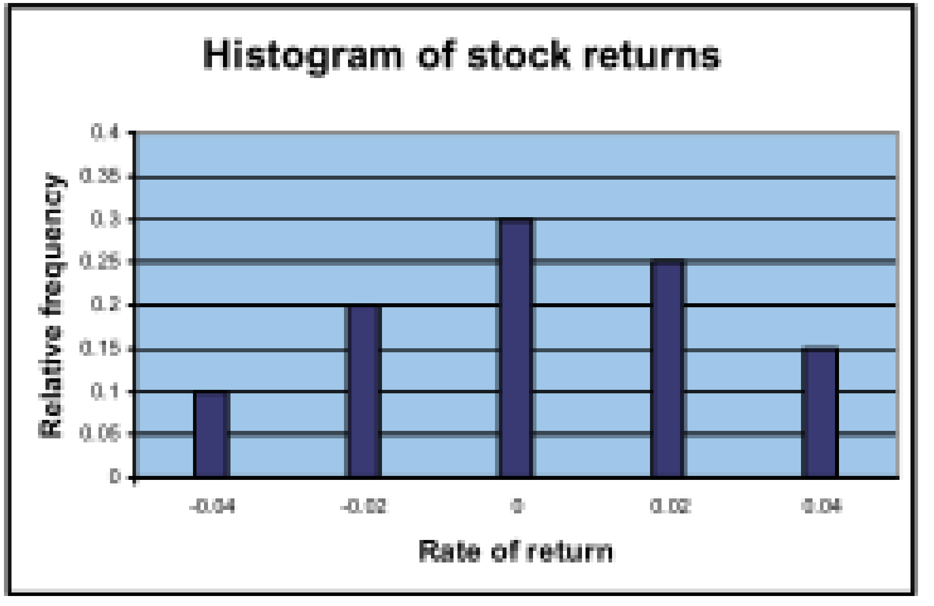
where $\tilde{A}_{i\cdot} \equiv \sqrt{p_i}A_{i\cdot}$ and $\tilde{b}_i \equiv \sqrt{p_i}b_i$

Working out the minimal ESRE hedge

- Proof
  - Transform ESRE into SSRE

Example

- Exercise 2.7 p.54
  - Risk-free rate 0%
  - Find optimal hedge of exposure X
- Share price £2
  - Share pay-off = (1+rate of return) x initial investment
  - Evaluate the cost of the best hedge



PFCo rate of return	Amount X to be hedged
0.04	£2000
0.02	£1000
0	£100
-0.02	£20
-0.04	£0