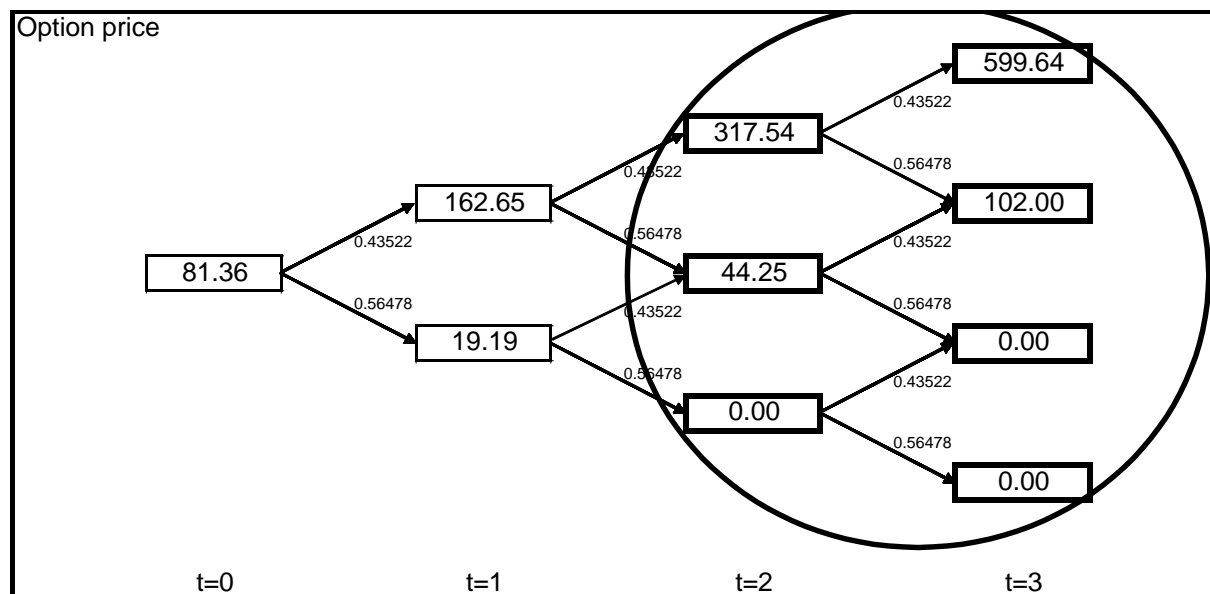


Lecture 3

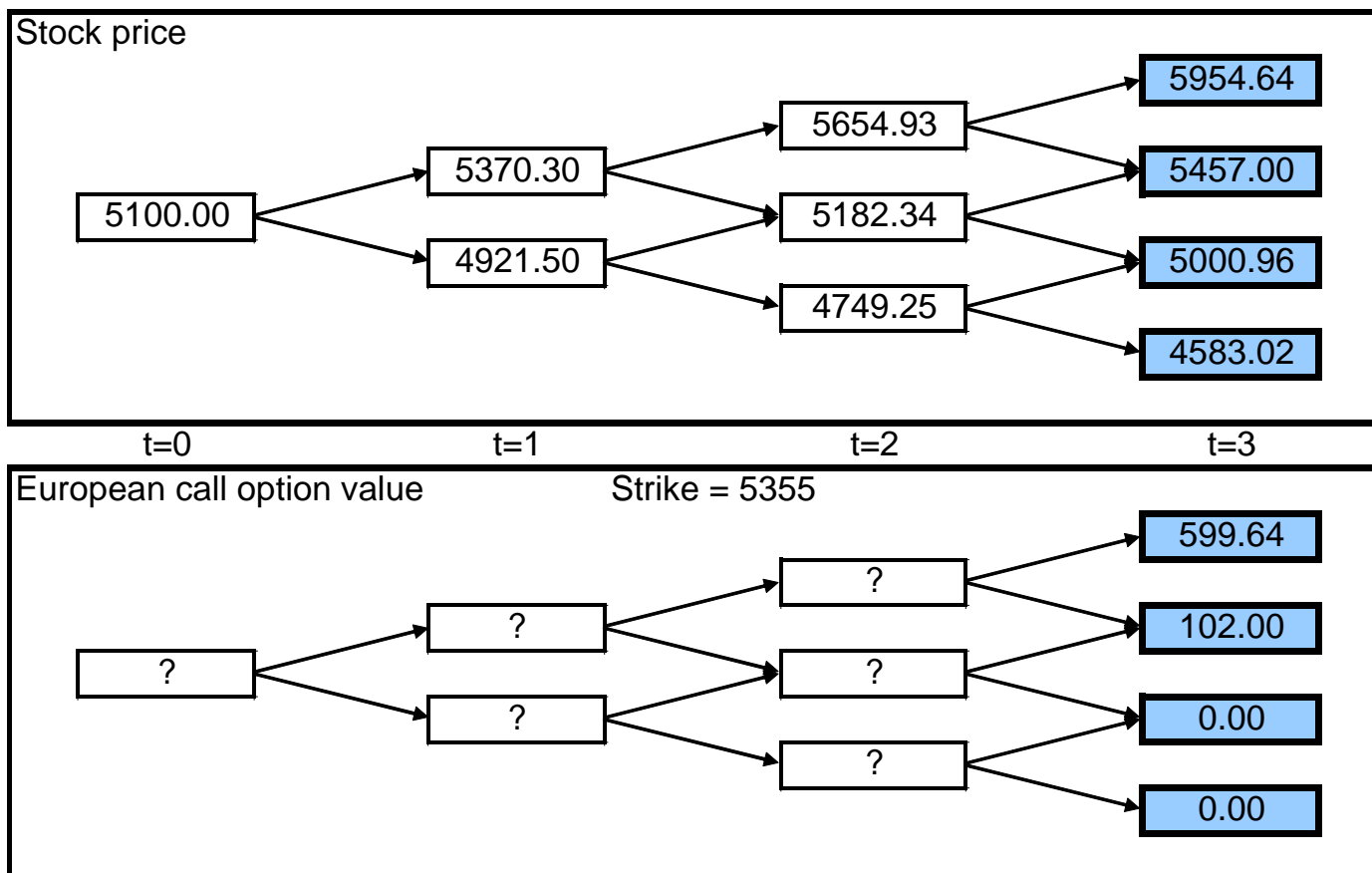
Risk-neutral pricing
in dynamically
complete markets
and continuous
time model



Risk-neutral pricing in dynamically complete markets



Summary of complete market dynamic hedging



Summary of complete market dynamic hedging

Stock prices			
			5954.64
		5654.93	
	5370.30		5457.00
5100.00		5182.34	
	4921.50		5000.96
		4749.25	
			4583.02
t=0	t=1	t=2	t=3

Value of the replicating portfolio			
			599.64
		317.54	
	162.66		102.00
81.36		44.25	
	19.19		0.00
		0.00	
			0.00
t=0	t=1	t=2	t=3

Bank account			
			599.64
		-5337.39	
	-2942.92		102.00
-1548.87		-1114.88	
	-483.63		0.00
		0.00	
			0.00

Number of shares			
			0.000
		1.000	
	0.578		0.000
0.320		0.224	
	0.102		0.000
		0.000	
			0.000

Risk-neutral probabilities in valuation

- Option value = value of self-financing replicating portfolio (bank account, delta)
- Option value can be calculated without knowing the delta!

$$q_u R_u + q_d R_d = R_f$$

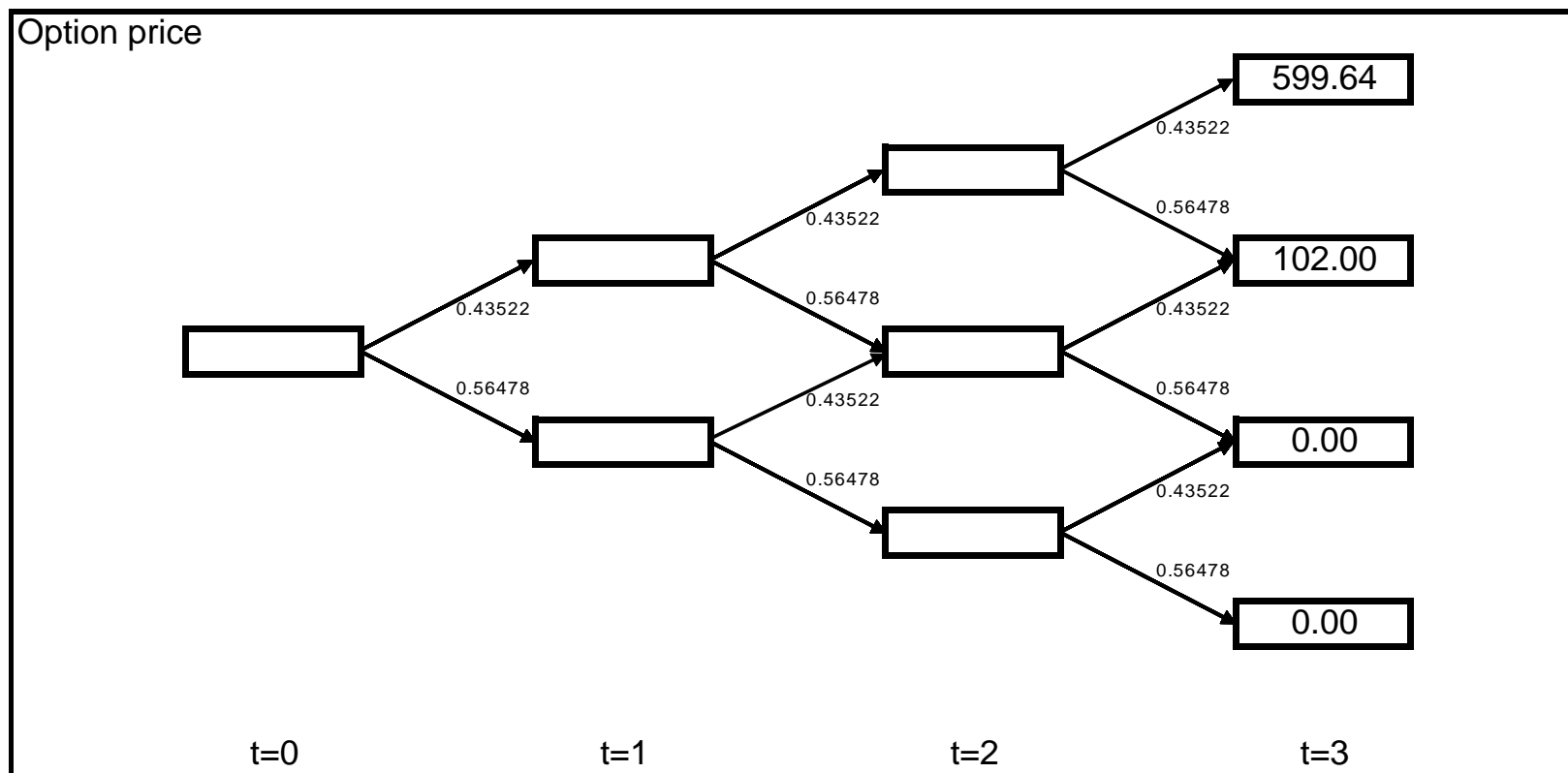
$$q_u + q_d = 1$$

$$q_u = 0.435\ 23 \quad q_d = 0.564\ 77$$

$$\text{value}(C) = (q_u C_u + q_d C_d) / R_f$$

- Convenient but effectively the same as hedging with stock and bond, see equation (5.11)

Fill in the option prices ($R_f = 1.0033$)



Objective versus risk-neutral probabilities

- Objective denoted P
- Risk-neutral denoted Q
- The meaning of objective probability
 - Probability of stock market movement
 - 3 high returns in a row = $p_u \times p_u \times p_u = 0.125$
- The meaning of risk-neutral probability
 - Cost of insurance against a particular stock market scenario
 - 3 high returns in a row = $q_u \times q_u \times q_u / (R_f)^3 = 0.082$
- More on this much later (Chapter 9)

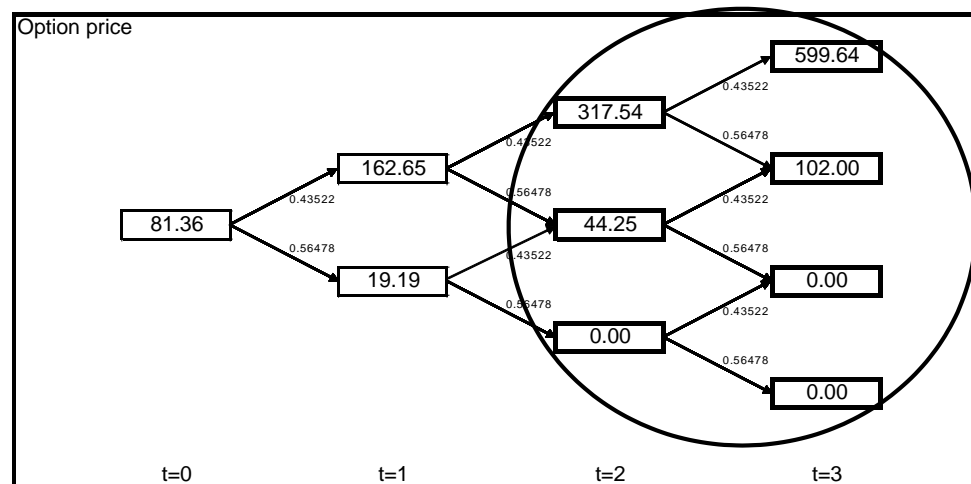
Option price as a conditional expectation

- One-period pricing:

$$\text{no - arbitrage value}(C) = \frac{0.43523C_u + 0.56477C_d}{1.0033}$$

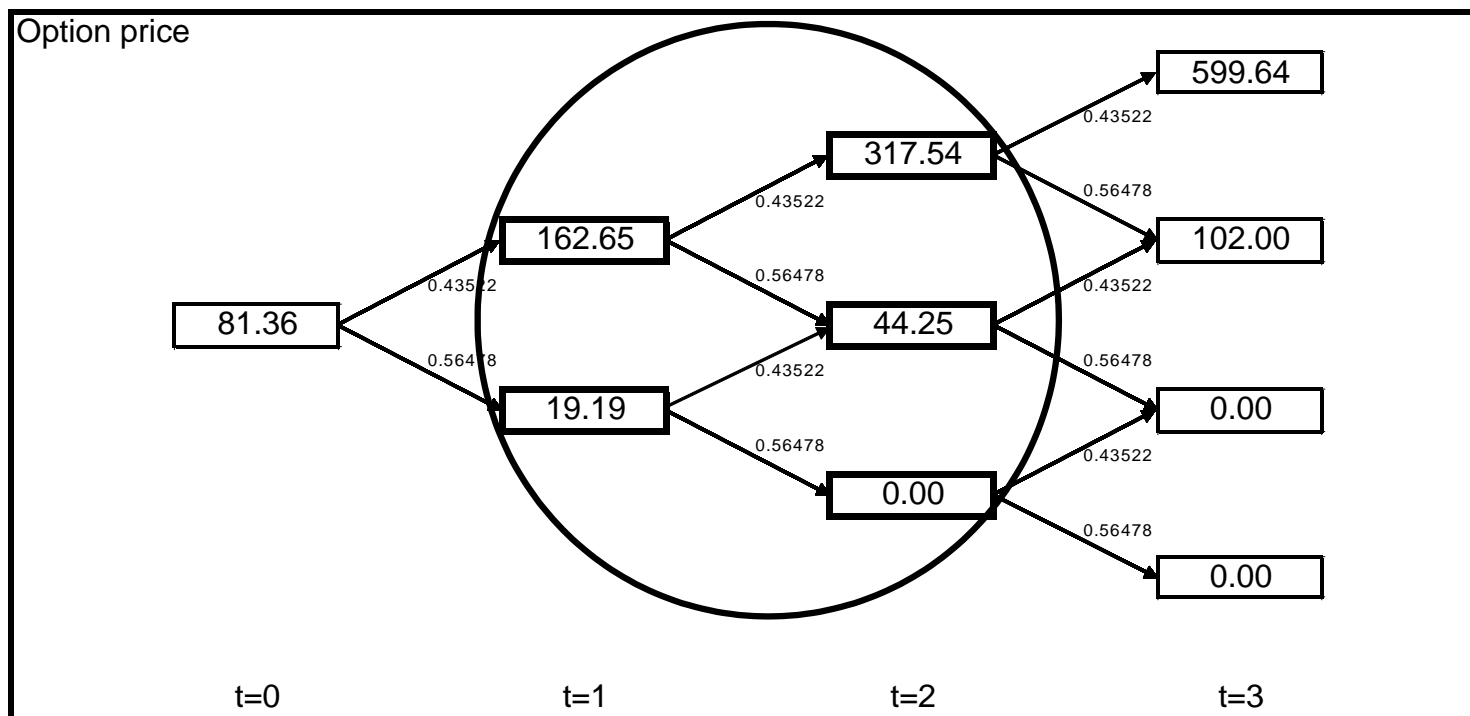
- More compact notation:

$$C_2 = E_2^Q \left[\frac{C_3}{R_f} \right]$$



Option price as a conditional expectation

$$C_1 = E_1^Q \left[\frac{C_2}{R_f} \right]$$



Option price as a conditional expectation

- Substituting recursively

$$C_0 = E_0^Q \left[\frac{1}{R_f} E_1^Q \left[\frac{1}{R_f} E_2^Q \left[\frac{C_3}{R_f} \right] \right] \right] = \frac{1}{R_f^3} E_0^Q \left[E_1^Q \left[E_2^Q [C_3] \right] \right]$$

- Law of iterated expectations: in a chain of conditional expectations the one with the lowest time index prevails

$$C_0 = \frac{1}{R_f^3} E_0^Q [C_3]$$

Law of iterated expectations: “Proof”

- We want to show

$$\frac{1}{R_f^3} E_0^Q [E_1^Q [E_2^Q [C_3]]] = \frac{1}{R_f^3} E_0^Q [C_3]$$

- The left-hand side equals 81.36
- Let us evaluate the unconditional expectation on the right-hand side

$$E_0^Q [C_3]$$

Law of iterated expectations: “Proof”

$$E_0^Q [C_3]$$

- Evaluate unconditional risk-neutral probability of reaching individual values of C_3
- Unconditional probability of a path = product of one-step conditional probabilities on that path
- $Q(C_3 = 599.64) = q_u \times q_u \times q_u = 0.082\ 44$
- $Q(C_3 = 102.00) = 3\ q_u \times q_u \times q_d = 0.320\ 94$
- $Q(C_3 = 0.00)$
 $= 1 - (0.082\ 44 + 0.320\ 94) = 0.596\ 62$

Law of iterated expectations: “Proof”

$$\frac{1}{R_f^3} E_0^Q [C_3] =$$

Summary of risk-neutral pricing

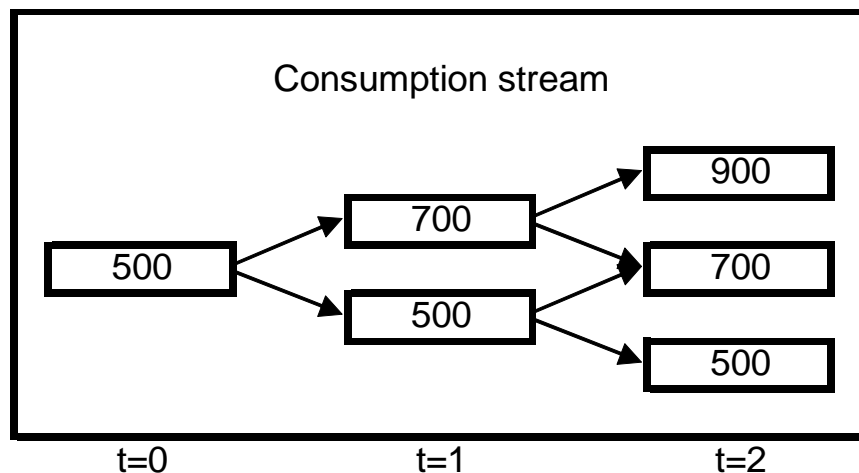
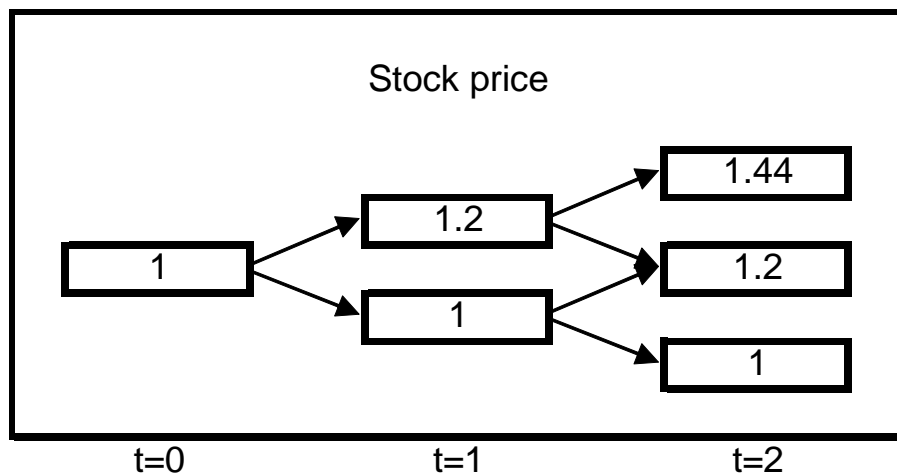
- The principle of dynamic asset pricing goes far beyond the simple call option pricing example
 - Exotic options
 - Fixed income securities
 - Investment projects (real options)
 - Firm valuation (see Copeland&Weston, corporate finance textbook)

Summary of risk-neutral pricing

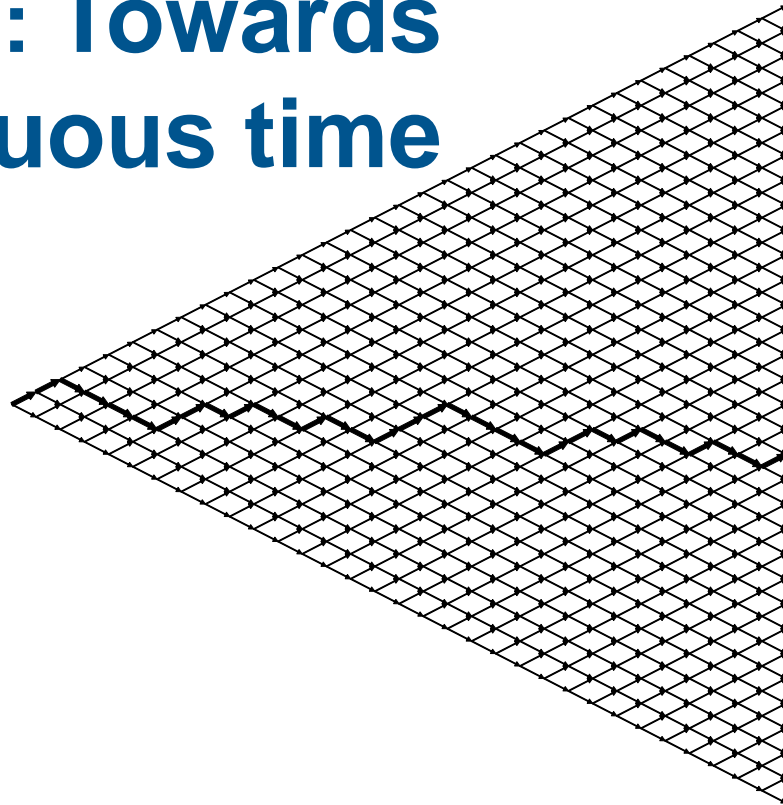
- Set up a model for basis assets
- Check that the model is complete (number of basis assets equals number of one-period scenarios)
- Calculate risk-neutral probabilities in each one-period model
- Evaluate the cash flow of the focus asset (European option can only be exercised at maturity => cash flow only at maturity)
- Price the cash flow using risk-neutral probs, working backwards from terminal date

Risk-neutral pricing with cash flow at multiple dates

- Exercise
 - Risk-free rate 5%
 - IID stock returns
 - How much money is needed to finance the consumption stream?



Chapter 6: Towards continuous time



Towards continuous time

- Time to maturity fixed
- Rehedging interval $\rightarrow 0$
- Number of periods $\rightarrow \infty$
- Two types of limit
 - Brownian motion limit \rightarrow Itô processes (Chapter 10, 11)
 - Poisson jump limit \rightarrow Lévy processes (not required)

Model calibration

- Assume IID returns over time interval Δt
- What is the mean and variance of returns over time T ?
- For every Δt choose model parameters so that mean and variance over time T remain constant

Properties of IID variables

Facts:

- For *any* collection of random variables X_1, X_2, \dots, X_n

$$\begin{aligned} E[X_1 + X_2 + \dots + X_n] &= E[X_1] + E[X_2] + \dots + E[X_n], \\ \text{expectation of a sum} &= \text{sum of expectations.} \end{aligned}$$

- If X_1, X_2, \dots, X_n are *independent* random variables then for *any* functions f_1, f_2, \dots, f_n the random variables $f_1(X_1), f_2(X_2), \dots, f_n(X_n)$ are again *independent*.

- Independent random variables are uncorrelated (however uncorrelated r.v. need not be independent).

- For *uncorrelated* random variables X_1, X_2, \dots, X_n we have

$$\begin{aligned} \text{Var}(X_1 + X_2 + \dots + X_n) &= \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n), \\ \text{variance of a sum} &= \text{sum of variances.} \end{aligned}$$

Time scaling of mean and variance

- Divide 1 hour into 12 five-minute intervals

$$R_{hour} = R_1(5 \text{ min}) R_2(5 \text{ min}) \dots R_{12}(5 \text{ min}).$$

- Logarithm of product = sum of logarithms

$$\ln R_{hour} = \ln R_1(5 \text{ min}) + \ln R_2(5 \text{ min}) + \dots + \ln R_{12}(5 \text{ min}).$$

- R_i independent $\Leftrightarrow \ln R_i$ independent
- The mean and variance of log returns grow linearly with time horizon

$$\begin{aligned} E[\ln R_{hour}] &= 12E[\ln R(5 \text{ min})] \\ \text{Var}(\ln R_{hour}) &= 12\text{Var}(\ln R(5 \text{ min})). \end{aligned}$$

- 1 hour has 60 minutes
- 1 trading day only has 8 (not 24) hours
- Trading week has 5 (not 7) days
- Prices do not move (very much) when markets are closed =>
- Mean and variance of log returns increase linearly with trading time (not calendar time)

IID returns and log returns

- The mean and variance of *log returns* grow linearly with trading time
- On very short time horizons rates of return are small; from Taylor expansion

$$\ln R = R - 1$$

- For short time horizons mean and variance of *returns* grow linearly with time
- ‘Square root law for volatility of returns’ observed at Paris Bourse as early as 1863

$$\text{Std}(X) = \sqrt{\text{Var}(X)}$$

Towards Brownian motion

- Linear scaling of mean and variance applies both to
 - Brownian motion limit
 - Poisson jump limit
- We are going to specialize our results to Brownian motion limit
 - Keep probabilities constant
 - Change the size of returns

Properties of mean and variance

- Let X be a random variable, and a, b constants. Then

$$\begin{aligned}E[a + bX] &= a + bE[X] \\ \text{Var}(a + bX) &= \text{Var}(bX) = b^2 \text{Var}(X)\end{aligned}$$

Constructing model of daily returns

- Must find a way to scale mean and variance separately
- Decompose $\ln R(\text{month})$ into two parts

$$\ln R_{\text{month}} = \underbrace{E[\ln R_{\text{month}}]}_{\text{non-random part}} + \underbrace{(\ln R_{\text{month}} - E[\ln R_{\text{month}}])}_{\text{random part with mean zero}}$$

- Scale the first by $1/21$, second by $\sqrt{1/21}$

$$\ln R_{\text{day}} = \frac{1}{21} E[\ln R_{\text{month}}] + \sqrt{\frac{1}{21}} (\ln R_{\text{month}} - E[\ln R_{\text{month}}])$$

- Same scaling applies to risk-free return

$$\ln R_f(\text{day}) = \frac{1}{21} \ln R_f(\text{month})$$

Constructing model of daily returns

$$\begin{aligned}R_u(\text{month}) &= 1.053 \text{ with } p_u = \frac{1}{2} \\R_d(\text{month}) &= 0.965 \text{ with } p_d = \frac{1}{2} \\R_f(\text{month}) &= 1.0033\end{aligned}$$

$$\begin{aligned}E[\ln R_{\text{month}}] &= p_u \ln R_u(\text{month}) + p_d \ln R_d(\text{month}) = \\&= \frac{0.0516}{2} - \frac{0.0356}{2} = 0.008\end{aligned}$$

$$\ln R_u(\text{day}) = \frac{0.008}{21} + \sqrt{\frac{1}{21}} (0.0516 - 0.008) = 0.00990$$

$$\ln R_d(\text{day}) = \frac{0.008}{21} + \sqrt{\frac{1}{21}} (-0.0356 - 0.008) = -0.00913$$

$$\ln R_f(\text{day}) = \frac{1}{21} \ln(1.0033) = 1.6 \times 10^{-4}.$$

Numerical implementation of binomial pricing

- Calculate returns from log returns

$$R_u(\text{day}) = e^{\ln R_u(\text{day})} = e^{0.00990} = 1.00995$$

$$R_d(\text{day}) = e^{\ln R_d(\text{day})} = e^{-0.00913} = 0.99091$$

$$R_f(\text{day}) = R_f(\text{month})^{1/21} = 1.0033^{\frac{1}{21}} = 1.00016$$

- Find risk-neutral probabilities

$$q_u(\text{day}) = \frac{R_f(\text{day}) - R_d(\text{day})}{R_u(\text{day}) - R_d(\text{day})} = \frac{1.00016 - 0.99091}{1.00995 - 0.99091} = 0.486$$

$$q_d(\text{day}) = 0.514$$

- Price options as usual