

University of Rome Tor Vergata
Financial Market Models Coursework 1

1. **(25 points) Option Greeks.** The Black-Scholes formula for the price of a European call option is

$$C(S, K, r, \sigma, \tau) = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \quad (1)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad (2)$$

$$d_2 = d_1 - \sigma\sqrt{\tau}, \quad (3)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt. \quad (4)$$

Find the *Theta*, *Vega* and *Rho* of the call option. The *Theta*, *Vega* and *Rho* of the call option are defined as the rate of change of the value of the option with respect to the time to maturity (τ), volatility of the underlying stock (σ) and interest rate (r), respectively.

2. **(25 points) Taylor Expansion.**

Consider the Black-Scholes formula from question 1.

- (a) Find how the price reacts to a small change in the underlying asset price S using a second-order Taylor series expansion.
- (b) Assume that we observed $S = 305$, $K = 300$, $r = 0.08$, $\sigma = 0.25$ and $\tau = 4/12$. If today the underlying asset price increases to 320, what is the new option price implied by the Black-Scholes formula and Taylor expansion from question (a)?
- (c) How big is the difference between the Black-Scholes price and the price approximated by the Taylor expansion in part (b)? Do you think that the difference would be smaller or bigger if we were to use a first-order Taylor expansion? In this case, when would the first and second-order Taylor approximations give a similar result?

3. **(10 points) Hedging in Complete and Incomplete Markets.** Explain what complete and incomplete markets are and discuss their hedging implications.

4. **(10 points) Hedging Problem.** An investment bank believes that the stock return R can take on three values, $x_1 = 1.3$, $x_2 = 1.1$ and $x_3 = 0.8$ with probabilities $p_1 = 0.3$, $p_2 = 0.5$ and $p_3 = 0.2$, respectively. In addition, there is a risk-free bank account with rate of return $r_f = 0.05$ on both borrowing and lending. The investment bank has sold one unit of a digital put option with payoff D :

$$D = 0 \text{ for } R > 1.05 \quad (5)$$

$$D = 1 \text{ for } R \leq 1.05. \quad (6)$$

The bank wishes to hedge its exposure by constructing a hedging (replicating) portfolio using the stock and the risk-free bank account.

- (a) Find the hedge that minimizes the expected squared replication error. State clearly how much money one has to deposit in (or borrow from) the bank account and what is the value of the stock that one has to buy (sell).
 - (b) Write down the cost of the optimal hedging portfolio computed in part (a).
5. **(5 points) Arbitrage.** Explain the concept of arbitrage in general and how it relates to state prices. What are the differences between type I and type II arbitrage?
6. **(25 points) Binomial Model - Call Option .** Consider a two-period (dates 0, 1, and 2) binomial model of an economy in which a stock and a bond are traded. Let the risk-free return be $R_f = 1.25$ in every period. Assume that the stock price at time $t = 0$ is 4, and in every period the return is $R_u = 2$ or $R_d = \frac{1}{2}$. What is the time 0 no-arbitrage price of a European call option, if the strike price is 6 and the expiration date is $t = 2$?