# LIFE INSURANCE
# MATLAB/R EXERCISES

PROF. KATIA COLANERI

## Task 1

The goal of this task is to compute the net single premium of a unit linked life pure endowment contract.

We begin with the description of the model. We consider a financial market which, under the risk neutral probability $\mathbf{Q}$ has the following dynamics:

- One riskless asset with price process $dB_t = B_t\ r\ dt$ and $B_0 = 1$, where $r$ is the risk free interest rate;
- One risky asset with price process $dS_t = S_t\ r\ dt + \sigma_S dW_t^S$, and $S_0 = s$ where $(W_t^S)_{t\geq 0}$ is a Brownian motion.

We complement the financial market with an insured life with remaining lifetime $T_x$. We also assume that the mortality intensity is constant and equal to $\mu_x$. Then ${}_tp_x = e^{-\mu_x t}$ for all $t$.

We consider a unit linked pure endowment contract with maturity $T$ and sum insured given by $\phi(S_T) = \max(S_0 e^{g_1 T}, S_T)$, where $g_1$ is the minimum guaranteed rate.

The task consists of computing the net single premium of the contract $\Pi_0 = \mathbb{E}^J\left[\phi(S_T)e^{-rT}e^{-\mu T}\right]$, where $J = \mathbf{Q} \times \widetilde{\mathbf{P}}$.

To do that, we want to apply Montecarlo, according to the following steps:

(1) Generate $N = 1000$ trajectories of the stock price process from $t = 0$ to $t = T$, using Euler discretization scheme
(2) For each trajectory compute the present value of the benefit given by $\phi(S_T)e^{-rT}e^{-\mu T}$
(3) Average over all trajectory to compute the single premium $\Pi_0$

## Task 2

For this second task we consider a different financial market model and a different mortality intensity. We assume that under the risk neutral probability $\mathbf{Q}$ the financial market has the following dynamics:

- One riskless asset with price process $dB_t = B_t\ r_t\ dt$ and $B_0 = 1$
- $r_t$ is the short rate that has the dynamics: $dr_t = a(\bar{r} - r_t)dt + b dW_t^r$, and $r_0 \in \mathbb{R}$, where $(W_t^r)_{t\geq 0}$ is a Brownian motion

Katia Colaneri, Department of Economics and Finance, University of Rome Tor Vergata, Via Columbia 2, 00133 Rome, Italy.

*E-mail address*: katia.colaneri@uniroma2.it.

- One risky asset with price process $dS_t = S_t\ r\ dt + \sigma_S dW_t^S$, and $S_0 = s$ where $(W_t^S)_{t\geq 0}$ is a Brownian motion.

We complement the financial market with an insured life with remaining lifetime $T_x$. We assume that the mortality intensity is stochastic with dynamics: $d\mu_t = \kappa(\bar{\mu} - \mu_t)dt + \eta\sqrt{\mu_t}dW_t^\mu$, and $\mu_0 \in \mathbb{R}^+$, where $(W_t^\mu)_{t\geq 0}$ is a Brownian motion. Recall that in this case $\widetilde{\mathbf{P}}(T_x > T) = \mathbb{E}^{\widetilde{\mathbf{P}}}\left[e^{-\int_0^T \mu_u du}\right]$.

All Brownian motions are assumed to be independent.

We consider a unit linked pure endowment contract with maturity $T$ and sum insured given by $\phi(S_T) = \max(S_0 e^{g_1 T}, S_T)$, where $g_1$ is the minimum guaranteed rate.

The task consists of computing the net single premium of the contract $\Pi_0 = \mathbb{E}^J\left[\phi(S_T)e^{-\int_0^T r_u du}e^{-\int_0^T \mu_u du}\right]$, where $J = \mathbf{Q} \times \widetilde{\mathbf{P}}$.

To do that, we want to apply Montecarlo, according to the following steps:

(1) Generate $N = 1000$ trajectories of the stock price process, the interest rate process, and the mortality process the from $t = 0$ to $t = T$, using Euler discretisation scheme
(2) For each trajectory compute the present value of the benefit given by $\phi(S_T)e^{-\int_0^T r_u du}e^{-\int_0^T \mu_u du}$ (use an appropriate function to compute the integral, for example cumsum)
(3) Average over all trajectory to compute the single premium $\Pi_0$

## Parameters

Feel free to play with parameters: these are just suggested values.

(1) Parameters for the Stock price:
  - $S_0 = 100$
  - $\sigma = 0.1$
(2) Parameters for the short rate:
  - $a = 2$
  - $b = 0.05$
  - $\bar{r} = 0.03$
  - $r_0 = 0.07$
(3) Parameters for the mortality intensity:
  - $\kappa = 2$
  - $\eta = 0.03$
  - $\bar{u} = 0.001$
  - $u_0 = 0.004$
(4) Other parameters
  - $T = 10$
  - $g_1 = 0.02$

## Appendix: Euler Discretisation scheme - Matlab version

**Algorithm 1** N trajectories of Stock price, short rate and mortality intensity

**Require:** $N = 10000$ ▷ Number of trajectories
**Require:** $M = 100$ ▷ Number of time points in 1 year
$dt = 1/M$ ▷ Fix a time step
$rng('default')$ ▷ Set seed for reproducibility. In R you may fix set.seed(123)
$dW_S \leftarrow sqrt(dt) \cdot randn(N, M \cdot T)$ ▷ Generate a Brownian Motion for the stock
$dW_r \leftarrow sqrt(dt) \cdot randn(N, M \cdot T)$ ▷ Generate a Brownian Motion for the short rate
$dW_S \leftarrow sqrt(dt) \cdot randn(N, M \cdot T)$ ▷ Generate a Brownian Motion for the mortality intensity
$S \leftarrow zeros(N, M \cdot T + 1)$ ▷ initialize a matrix $S$
$S(:, 1) \leftarrow S_0$
$r \leftarrow zeros(N, M \cdot T + 1)$ ▷ initialize a matrix $r$
$r(:, 1) \leftarrow r_0$
$u \leftarrow zeros(N, M \cdot T + 1)$ ▷ initialize a matrix $u$
$u(:, 1) \leftarrow u_0$
**for** $i = 1 : N$ **do**
  **for** $j = 1 : M \cdot T + 1$ **do**
    $S(i, j+1) \leftarrow S(i, j) + S(i, j) \cdot r(i, j) \cdot dt + S(i, j) \cdot \sigma \cdot dW_S(i, j)$
    $r(i, j+1) \leftarrow r(i, j) + a \cdot (\bar{r} - r(i, j)) \cdot dt + b \cdot dW_r(i, j)$
    $u(i, j+1) \leftarrow u(i, j) + \kappa(\bar{u} - u(i, j)) \cdot dt + \sqrt{u(i, j)} \cdot \eta \cdot dW_u(i, j)$
  **end for**
**end for**
$discount \leftarrow zeros(N, M \cdot T + 1)$ ▷ initialize a parameter for the double discount
$discount \leftarrow exp(-cumsum(r(:, 1 : end) + u(:, 1 : end), 2) \cdot dt)$ ▷ Compute the double discount process
$PV \leftarrow zeros(N, 1)$
$PV \leftarrow discount(:, end) \cdot max(S_0 \cdot e^{g_1 T}, S(:, end))$ ▷ Compute the present value of the benefit. Note that this is product of vectors
$Pi \leftarrow mean(PV)$ ▷ Single premium

## Appendix: Euler Discretisation scheme - R version

**Algorithm 2** N trajectories of Stock price, short rate and mortality intensity

**Require:** $N = 10000$ ▷ Number of trajectories
**Require:** $M = 100$ ▷ Number of time points in 1 year
$dt = 1/M$ ▷ Fix a time step
$set.seed(123)$ ▷ Set seed for reproducibility
$S \leftarrow matrix(0, nrow = N, ncol = M \cdot T + 1)$ ▷ initialize a matrix $S$
$S[,1] \leftarrow s_0$
$r \leftarrow matrix(0, nrow = N, ncol = M \cdot T + 1)$ ▷ initialize a matrix $r$
$r[,1] \leftarrow r_0$
$u \leftarrow matrix(0, nrow = N, ncol = M \cdot T + 1)$ ▷ initialize a matrix $u$
$u[,1] \leftarrow u_0$
$dW_S < -matrix(rnorm(N \cdot M \cdot T, 0, sqrt(dt)), nrow = N)$ ▷ Generate a Brownian Motion for the stock
$dW_r < -matrix(rnorm(N \cdot M \cdot T, 0, sqrt(dt)), nrow = N)$ ▷ Generate a Brownian Motion for the short rate
$dW_u < -matrix(rnorm(N \cdot M \cdot T, 0, sqrt(dt)), nrow = N)$ ▷ Generate a Brownian Motion for the mortality intensity
**for** $i\ in\ 1 : N$ **do**
  **for** $j\ in\ 1 : M \cdot T + 1$ **do**
    $S[i, j+1] \leftarrow S[i, j] + S[i, j \cdot r[i, j] \cdot dt + S[i, j] \cdot \sigma \cdot dW_S[i, j]$
    $r[i, j+1] \leftarrow r[i, j] + a \cdot [\bar{r} - r[i, j]] \cdot dt + b \cdot dW_r[i, j]$
    $u[i, j+1] \leftarrow u[i, j] + \kappa[\bar{u} - u[i, j]] \cdot dt + \sqrt{u[i, j]} \cdot \eta \cdot dW_u[i, j]$
  **end for**
**end for**
$discount \leftarrow matrix(0, nrow = N, ncol = M \cdot T + 1)$ ▷ initialize a parameter for the double discount
$discount \leftarrow exp(-cumsum(r[\ , 1 : (M \cdot T + 1)] + u[\ , 1 : (M \cdot T + 1)]) \cdot dt)$ ▷ Compute the double discount process
$PV \leftarrow matrix(0, nrow = N, ncol = 1)$
$PV \leftarrow discount[\ , M \cdot T + 1] \cdot max(S_0 \cdot e^{g_1 T}, S[\ , M \cdot T + 1])$ ▷ Compute the present value of the benefit. Note that this is product of vectors
$Pi \leftarrow mean(PV)$ ▷ Single premium